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Mathematics 145A, Winter 2016, Examination 2

Answer Key

1. [25 *points*] (i) Suppose that X is a topological space and $A \subset X$ is not closed in X. Explain why A is properly contained in its closure $\overline{A}$ (in X).

(ii) Suppose that $(X_1, \mathcal{U}_1)$ and $(X_2, \mathcal{U}_2)$ are topological spaces. Describe the product topology on $X_1 \times X_2$ in terms of the open subsets in $\mathcal{U}_1$ and $\mathcal{U}_2$.

SOLUTION

(i) By definition the closure C of a subset A is a closed set containing A, and if A is not closed then C cannot be equal to A. Since $A \subset C$ but $A \neq C$, this means that A must be properly contained in C.■

(ii) The product topology is generated by all subsets of the form $V_1 \times V_2$, where V_1 is an open subset in X_1 and V_2 is an open subset in X_2. In fact, it is the family of all subsets which are unions of such products.■

2. [*25 points*] Suppose that $X \subset \mathbb{R}^2$ is a union $A \cup B$ where

$$A = [-2,2] \times [-1,1] \, , \qquad B = [-1,1] \times [-2,2] \, .$$

Prove that X is arcwise connected. [*Hint:* Sketch the region in the coordinate plane.]

SOLUTION

First of all, by construction each of A and B is a product of two arcwise connected subspaces, and hence each is arcwise connected. Furthermore, the intersection $A \cap B$ is nonempty because it is $[-1,1] \times [-1,1]$. Since the union of two arcwise connected subspaces is arcwise connected if their intersection is nonempty, it follows that $A \cup B$ is also arcwise connected.∎

3. [*25 points*] Suppose that the topological space X is the union of two subsets $A \cup B$ such that both A and B are compact. Prove that $A \cup B$ is also compact. [*Hint:* Given an open covering $\mathcal{U}$ of X, consider the induced open coverings of A and B.]

SOLUTION

Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of X, and for each α let $V_\alpha = U_\alpha \cap A$ and $W_\alpha = U_\alpha \cap B$. Consider the associated open coverings $\mathcal{V} = \{V_\alpha\}$ of A and $\mathcal{W} = \{W_\alpha\}$ of B. Since both A and B are compact there are finite subcoverings $\{V_{\alpha(1)}, \cdots, V_{\alpha(k)}\}$ and $\{W_{\alpha(k+1)}, \cdots, W_{\alpha(k+m)}\}$ of A and B respectively. By the definitions of the sets in the subcoverings we have $A \subset U_{\alpha(1)} \cup \cdots \cup U_{\alpha(k)}$ and $B \subset U_{\alpha(k+1)} \cup \cdots \cup U_{\alpha(k+m)}$. Since $X = A \cup B$, it follows that $\{U_{\alpha(1)}, \cdots, U_{\alpha(k+m)}\}$ is a finite subcovering of X. Since $\mathcal{U}$ was an arbitrary open covering, this means that X must be compact.■

4. [*25 points*] (i) Give an example of a nonmetrizable space (in other words a topological space $(X, \mathcal{U})$ which is not the underlying topological space for some metric space (X, d)).

(ii) State which of the following statements is/are true and which is/are false. Reasons are not needed for correct answers, but for incorrect answers they may yield partial credit.

The interval $(\frac{1}{2}, 1]$ is open in $\mathbb{R}$.

The interval $(\frac{1}{2}, 1]$ is open in $(0, 1]$ with respect to the subspace topology.

The interval $(\frac{1}{2}, 1]$ is closed in $\mathbb{R}$.

The interval $(\frac{1}{2}, 1]$ is closed $(0, 1]$ in with respect to the subspace topology.

SOLUTION

(i) There are many possibilities; perhaps the two simplest ones are the indiscrete topology on a set with more than one element and the finitary topology on an infinite set.■

(ii) We shall consider the statements in the order in which they are listed.

The first statement is FALSE because $1 \in \left(\frac{1}{2}, 1\right]$ and $\left(\frac{1}{2}, 1\right]$ does not contain an open interval of the form $(1 - \delta, 1 + \delta)$.

The second statement is TRUE because $\left(\frac{1}{2}, 1\right] = (0, 1] \cap \left(\frac{1}{2}, \frac{3}{2}\right)$ and the right hand side is an open subset in the subspace topology.

The third statement is FALSE because $\frac{1}{2}$ is a limit point of $\left(\frac{1}{2}, 1\right]$ in $\mathbb{R}$ but does not belong to that interval.

The fourth statement is FALSE because $\frac{1}{2}$ is a limit point of $\left(\frac{1}{2}, 1\right]$ in $(0, 1]$ but does not belong to $(0, 1]$.■