Mathematics 145A, Winter 2017, Examination 2

Answer Key

1. [20 *points*] Suppose that (X, d) is a metric space, let $a \in X$, let $\delta > 0$, and let $W_\delta(a)$ be the set of all $x \in X$ such that $d(a, x) > \delta$. Prove that $W_\delta(a)$ is an open subset of X. [*Hint:* The Triangle Inequality implies that $d(x, y) \geq |d(x, z) - d(y, z)|$; you may use this without proof.]

SOLUTION

Suppose that $y \in W_\delta(a)$, so that $d(a, y) = \delta + h$ for some $h > 0$. Therefore if $d(x, y) < h$ then we have $d(x, a) \geq d(x, y) - d(a, y) = \delta + h - d(a, y) > \delta + h - h > \delta$, which means that $x \in W_\delta(a)$. Therefore $W_\delta(a)$ is open.■

2. [30 *points*] (a) Show that the (countably infinite) intersection of the open intervals $\left(-\frac{1}{n}, 1\right)$ is not an open subset of the real line.

(b) A subset A of a metric space is said to be a G_δ **set** if it is a countable intersection $\cap_n V_n$ of open subsets V_n. Show that if $f : (X, d_X) \to (Y, d_Y)$ is continuous and $A \subset Y$ is a G_δ set, then so is $f^{-1}[A]$.

SOLUTION

(a) The intersection is the set of points t such that $t < 1$ and $t \geq -\frac{1}{n}$ for all n. The second conditions are equivalent to the inequality $t \geq 0$. Therefore the intersection is equal to $[0, 1)$, which is not an open subset.■

(b) Express $A = \cap_n V_n$ where each V_n is open in Y. Then we have

$$f^{-1}[A] \quad = \quad f^{-1}\left[\cap_n V_n\right] \quad = \quad \cap_n f^{-1}\left[V_n\right]$$

and hence $f^{-1}[A]$ is a G_δ set.■

3. [*25 points*] Recall that the taxicab separation d_T between two points on the coordinate plane $\mathbb{R}^2$ is defined as follows: If $p = (x_1, y_1)$ and $q = (x_2, y_2)$, then $d_T(p, q) = |x_2 - x_1| + |y_2 - y_1|$. Prove that d_T defines a metric on $\mathbb{R}^2$.

SOLUTION

The right hand side is a sum of two absolute values, and since each of the latter is nonnegative so is d_T.■

If $d_T(p, q) = 0$, then $0 = |x_2 - x_1| + |y_2 - y_1|$ implies that each of the two nonnegative summands is zero. But now $0 = |x_2 - x_1|$ and $0 = |y_2 - y_1|$ imply $x_1 = x_2$ and $y_1 = y_2$, which means that $p = q$.■

The function d_T is symmetric in p and q, for $|u - v| = |v - u|$ implies that

$$d_T(p, q) \quad = \quad |x_2 - x_1| + |y_2 - y_1| \quad = \quad |x_1 - x_2| + |y_1 - y_2| \quad = \quad d_T(q, p) \;.■$$

To prove the Triangle Inequality, first observe that $|c - a| \leq |c - b| + |b - a|$ for all $u, v \in \mathbb{R}$. If $r = (x_3, y_3)$ then

$$d_T(p, q) \quad = \quad |x_2 - x_1| \;+\; |y_2 - y_1| \quad \leq \quad (|x_2 - x_3| + |x_3 - x_1|) \;+\; (|y_2 - y_3| + |y_3 - y_1|) \quad =$$

$$(|x_2 - x_3| + |y_2 - y_3|) \;+\; (|x_3 - x_1| + |y_3 - y_1|) \quad = \quad d_T(p, r) \;+\; d_T(r, q)$$

which verifies the Triangle Inequality for d_T. We have now verified all of the conditions for d_T to be a metric.■

4. [25 *points*] (*a*) A map $f : (X, d_X) \to (Y, d_Y)$ of metric spaces is said to satisfy a *Lipschitz condition* if there is some constant $K_f > 0$ such that $d_Y(f(x), f(x')) \leq K_f \cdot d_X(x, x')$ for all $x, x' \in X$ (by one of the practice exercises we know that f is uniformly continuous). Prove that if f and $g : (Y, d_Y) \to (Z, d_Z)$ both satisfy Lipschitz conditions then so does their composite $g \circ f$.

(*b*) If X be a metric space, and let $T : X \times X \to X \times X$ be the coordinate transposition map sending (x, x') to (x', x). Prove that T is continuous with respect to a standard metric on $X \times X$. [*Hint:* It does not matter which of the product metrics d_1, d_2 or d_∞ is used.]

SOLUTION

(*a*) The hypotheses imply that there are constants $K_f > 0$ and $K_g > 0$ such that $d_Y(f(x), f(x')) \leq K_f \cdot d_X(x, x')$ for all $x, x' \in X$ and $d_Z(g(y), g(y')) \leq K_g \cdot d_Y(y, y')$ for all $y, y' \in Y$. Therefore we have

$$d_Z(g \circ f(x), g \circ f(x')) \quad \leq \quad K_g \cdot d_Y(f(x), f(x')) \quad \leq \quad K_g \cdot K_f \cdot d_X(x, x')$$

and hence $g \circ f$ also satisfies a Lipschitz condition.∎

(*b*) Let $p_1 : X \times X \to X$ and $p_2 : X \times X \to X$ denote projection onto the first and second factors respectively. Then we know that T is continuous if and only if each of the coordinate functions $p_1 \circ T$ and $p_2 \circ T$ are continuous. But $p_1 \circ T = p_2$ and $p_2 \circ T = p_1$; since the coordinate projection functions p_1 and p_2 are continuous, it follows that T is also continuous.∎