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# Mathematics 145A, Winter 2019, Examination 2

## Answer Key

**1.** [*25 points*] A subset of a topological space $X$ is said to be *locally closed* if it can be written as $F \cap V$ where $F$ is closed in $X$ and $V$ is open in $X$. Prove that every open subset is locally closed, and likewise for every closed subset. Give an example of a connected subset of $\mathbb{R}$ which locally closed but neither open nor closed. [*Hint:* The space $X$ is an open and closed subset of itself.]

SOLUTION

If $U$ is open then $U = X \cap U$ is locally closed because $X$ is closed in itself. If $F$ is closed then $F = X \cap F$ is locally closed because $X$ is open in itself. In $\mathbb{R}$ the half-open interval $[0, 1) = (-\infty, 1) \cap [0, \infty)$ where the first set is open and the second is closed, so $[0, 1)$ is locally closed. It is not closed because 1 is a limit point which is not in the set, and it is not open because no open interval of the form $\left(-\frac{1}{n}, \frac{1}{n}\right)$ is contained in it.■

**2.** [*25 points*] Suppose that the topological space $X$ is a union $A \cup B$ where both $A$ and $B$ are closed subsets of $X$, and let $f : X \to Y$ be a map of sets such that the restrictions $f|A$ and $f|B$ are continuous. Prove that $f$ itself must be continuous. [*Hint:* If $C$ is a subset of $X$, then $C = (C \cap A) \cup (C \cap B)$, and if $E \subset X$ then $(f|E)^{-1}[D] = E \cap f^{-1}[D]$.]

SOLUTION

We shall show that if $F$ is closed in $Y$ then $f^{-1}[F]$ is closed in $X$. The continuity hypotheses and the hints imply that

$$(f|A)^{-1}[F] = A \cap f^{-1}[F] \ , \qquad (f|B)^{-1}[F] = B \cap f^{-1}[F]$$

are both closed subsets of $X$. Since $X = A \cup B$ we have

$$f^{-1}[F] \ = \ \left(A \cap f^{-1}[F]\right) \cup \left(B \cap f^{-1}[F]\right) \ = \ (f|A)^{-1}[F] \cup (f|B)^{-1}[F]$$

is a union of two closed subsets and hence is also closed, proving the statement in the first sentence of this paragraph.■

**3.** [*25 points*] Using the definition for the boundary (frontier) of a subset, show that the boundary of both the open interval $(0,1) \subset \mathbb{R}$ and the closed interval $[0,1] \subset \mathbb{R}$ is the set $\{0,1\}$ of endpoints.

SOLUTION

By definition, $\text{Bdy}(A) = \text{Closure}(A) - \text{Interior}(A)$, so we have to descrive both of these sets when $A$ is the open unit interval. The closure is [0,1] because this is a closed set containing $(0,1)$, and each endpoint is a limit point (the endpoints are the limits of the sequences

$$\frac{1}{n}, \qquad \frac{n-1}{n}$$

as $n \to \infty$). The interior of $A$ is simply $A$ itself because $A$ is an open subset of $\mathbb{R}$. Therefore the boundary is the complement of $(0,1)$ in $[0,1]$, which is just the set $\{0,1\}$ of endpoints.∎

**4.** [*25 points*] Suppose that $X$ is a topological space and $X$ is a union $A \cup B$ where $A$ and $B$ are closed subsets of $X$. Prove that $X$ is $\mathbf{T}_1$ if both $A$ and $B$ are.

SOLUTION

Let $p \in X$. If $t \in A$, then by hypothesis $\{p\}$ is a closed subset of $A$ in the subspace topology, and since $A$ is a closed subset of $X$ it follows that $\{p\}$ is a closed subset of $X$. On the other hand, if $p \in B$ then similar reasoning applies with $B$ replacing $A$ throughout. Therefore $\{p\}$ is a closed subset of $X$ in all cases, which means that $X$ is $\mathbf{T}_1$.■

**5.** [*25 points*]

Let $A$ be an arbitrary subset of the unit circle $S^1 \subset \mathbb{R}^2$, which is all points $v = (x, y)$ such that $|v|^2 = x^2 + y^2 = 1$, and let $X_A \subset \mathbb{R}^2$ be the union of $A$ and the open disk defined by $x^2 + y^2 < 1$. Show that $X_A$ is arcwise connected (in fact, it is convex, but it is not necessary to show this).

SOLUTION

We know that the open disk of radiuis 1 is arcwise connected because every point $p$ in it can be joined to the origin by a closed curve of the form $(1 - t)p$, where $t \in [0, 1]$. We further claim that every point in $a \in A$ can be joined to the origin by a similar curve $(1 - t)a$, so the only thing to check is that the curve always stays inside $X_A$. This is true for $t = 0$ when the curve starts at $a$. But if $t > 0$ then $|a| = 1$ and $0 \leq (1 - t) < 1$ implies that $(1 - t)a$ lies in the open disk. Therefore the given curve joins $a \in S^1$ to the origin by a curve entirely within $X_A$.■

**6.** [*25 points*] Suppose that $A$ and $B$ are compact subsets of a metric space $X$. Show that $A \cap B$ is also compact.

SOLUTION

Let $\mathcal{U}$ be an open covering of $A \cup B$, and let $\mathcal{U}_A$ and $\mathcal{U}_B$ denote the intersections of open sets in $U$ with $A$ and $B$ respectively. Then the latter two are open coverings of $A$ and $B$ respectively and hence there are finite subcoverings of $A$ and $B$ having the forms

$$\mathcal{V}_A = \{U_{A,1} \cap A, \cdots, U_{A,m} \cap A\} , \quad \mathcal{V}_B = \{U_{B,1} \cap B, \cdots, U_{B,m} \cap B\}$$

respectively. It follows that the sets $U_{A,i}$ and $U_{B,j}$ combine to form a finite subcovering of $A \cup B$.■