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Mathematics 145A, Winter 2017, Examination 3

Answer Key

1. [25 *points*] Suppose that X is a topological space such that every nonempty open subset U contains infinitely many points. Prove that every point of X is a limit point of X.

SOLUTION

A point $p \in X$ is a limit point of X if and only if for every open subset U containing p we have

$$(U - \{p\}) \cap X \neq \emptyset$$

and since $U - \{p\} \subset X$ the latter is equivalent to the condition $U - \{p\} \neq \emptyset$.

Now let $p \in X$; to prove the assertion in the exercise we have to know that if U is an open subset of X with $p \in U$, then $U - \{p\}$ is nonempty. In fact, we are assuming that every open subset U is infinite, and therefore the same is true for $U - \{p\}$. As in the first paragraph, this implies that the (arbitrary) point p is a limit point of X.■

2. [30 *points*] Let X and Y be topological spaces, and let $A \subset X$ and $B \subset Y$ be nonempty subsets of X and Y (respectively) such that $A \times B$ is closed in $X \times Y$. Prove that A is closed in X and B is closed in Y. [*Hints:* Recall that the vertical and horizontal slices $\{x_0\} \times Y$ and $X \times \{y_0\}$ are homeomorphic to Y and X respectively by the maps sending (x_0, y) and (x, y_0) to y and x. Also, recall the definition of subspace topologies for vertical and horizontal slices.]

SOLUTION

We first prove that A is closed in X. Since A and B are nonempty, there is at least one point $x_0 \in A$ and $y_0 \in B$, and of course we have $(x_0, y_0) \in A \times B$.

The horizontal slice $X \times \{y_0\}$, with the subspace topology, is homeomorphic to X by the map sending (x, y_0) to x. Since $A \times B$ is known to be closed in $X \times Y$, it follows that

$$A \times \{y_0\} \quad = \quad (A \times B) \ \cap \ (X \times \{y_0\})$$

is closed in $X \times \{y_0\}$ with respect to the subspace topology. Since a homeomorphism sends closed sets to closed sets, it follows that A, which is the image of $A \times \{y_0\}$ under the homeomorphism described above, is a closed subset of Y.

We now prove that B is closed in Y. The vertical slice $\{x_0\} \times Y$, with the subspace topology, is homeomorphic to Y by the map sending (x_0, y) to y. Since $A \times B$ is known to be closed in $X \times Y$, it follows that

$$\{x_0\} \times B \quad = \quad (A \times B) \ \cap \ (\{x_0\} \times Y)$$

is closed in $\{x_0\} \times Y$ with respect to the subspace topology. Since a homeomorphism sends closed sets to closed sets, it follows that B, which is the image of $\{x_0\} \times B$ under the homeomorphism described above, is a closed subset of Y.∎

3. [*25 points*] Given a Hausdorff topological space X and two points $p, q \in X$, let $\mathcal{N}_p$ and $\mathcal{N}_q$ be the families of all open subsets in X containing p and q respectively. Explain why $\mathcal{N}_p$ and $\mathcal{N}_q$ distinct families of subsets of X.

SOLUTION

We need to show that there is some subset which is in $\mathcal{N}_p$ but not in $\mathcal{N}_q$ or vice versa.

Since x is Hausdorff, there are disjoint open subsets U and V containing p and q respectively. It follows that U is in $\mathcal{N}_p$ but not in $\mathcal{N}_q$ and V is in $\mathcal{N}_q$ but not in $\mathcal{N}_p$, so $\mathcal{N}_p$ and $\mathcal{N}_q$ must be distinct subfamilies of subsets in X.■

4. [20 *points*] Let $A \subset \mathbb{R}$ be a subset which is compact and connected. Show that A is a closed interval $[a, b]$ for some a and b. [*Hint:* What are characterizations for compact subsets of $\mathbb{R}$ and for connected subsets of $\mathbb{R}$?]

SOLUTION

Since A is a connected subset of the real line, it must be an interval of the form (a, b), $[a, b]$, $(a, b]$ or $[a, b)$, where the possibilities $a = -\infty$ or $b = +\infty$ if (respectively) a is not an endpoint or b is not an endpoint.

Since a compact subset is bounded, it follows that all the endpoint must be genuine (finite) real numbers.

Finally, since a compact subset is closed and the only bounded intervals which are closed in the real line are closed intervals of the form $[a, b]$, it follows that A must be a closed interval $[a, b]$ for suitable $a \leq b \in \mathbb{R}$.■