ADDITIONAL EXERCISES FOR

MATHEMATICS 145A — Part 5

Winter 2014

11. The Hausdorff condition

1. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that the graph Γ_f is closed in $\mathbb{R}^2$ but f is not continuous. [*Hint:* Set $f(x) = 1/x$ if $x \neq 0$.]

Note. Later in this course we shall show that if the topologies for X and Y are suitably restricted, then a function $f : X \to Y$ is continuous if (and only if) its graph is a closed subset of $X \times Y$. There are also results of this type in other branches of mathematics (for example, in functional analysis).

2. Given a topological space $(X, \mathcal{T})$, let $\mathcal{B}$ be a base for its topology. Prove that X is Hausdorff if and only if for each pair of distinct points $u \neq v$ in X there exist basic open sets $U, V \in \mathcal{B}$ such that $u \in U$, $v \in V$, and $U \cap V = \emptyset$.

3. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and that A and B are closed subsets of X and Y respectively. Prove that $A \times B$ is a closed subset of $X \times Y$ with the product topology. [*Hint:* Explain why $A \times B$ is the intersection of $\pi_X^{-1}[A]$ and $\pi_Y^{-1}[B]$ and use the characterization of continuity involving inverse images of closed subsets.]

12. Connected spaces

Some of these problems will use material about cardinal numbers from Mathematics 144. There is a brief summary of the necessary background material in the directory file `cardinals.pdf`.

1. Let (X, d) be a metric space which contains more than one point. prove that the cardinality of X is at least as large as the cardinality of $\mathbb{R}$. [*Hint:* If $x, y \in X$, show that for each a such that $0 \leq a \leq d(x, y)$ there is some point $z \in X$ such that $d(x, z) = a$.]

2. (i) Let $U \subset \mathbb{R}^2$ be the open square $(0,1)^2$, let S be a subset of $\mathbb{R}$, and let $A_S = U \cup S \times \{1\}$. Using Proposition 12.19 of Sutherland, show that A_S is connected.

(ii) In the setting of part (i), show that $S \neq T$ implies $A_S \neq A_T$, and using this prove that the cardinality of the number of connected subsets in $\mathbb{R}^2$ is equal to $|\mathcal{P}(\mathbb{R})|$, where $\mathcal{P}(X)$ denotes the set of all subsets of a set X.

3. Prove that the cardinality of the set of connected subsets of $\mathbb{R}$ is equal to $|\mathbb{R}|$, and explain why this implies that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. [*Hint:* Explain why every connected subset of $\mathbb{R}$ is an open, closed or half open interval, and explain why the number of such intervals is equal to $\mathbb{R}$.]

4. (*i*) A topological space is said to be *totally disconnected* if it has a base consisting of subsets which are both open and closed. Explain why a discrete space is totally disconnected, and show that the set of all points x in $\mathbb{R}$ such that $x = 0$ or $x = 1/n$ for some positive integer n is a totally disconnected space which is not (homeomorphic to) a discrete space. [*Hint:* A subset of a discrete space has no limit points.]

(*ii*) Prove that the rational numbers (with the subspace topology inherited from $\mathbb{R}$) is a totally disconnected space in which every point is a limit point. [*Hint:* If q is a rational number, explain why for each positive integer n the sets

$$\left(q - \tfrac{1}{n}\sqrt{2},\ \ q + \tfrac{1}{n}\sqrt{2}\right) \cap \mathbb{Q} \qquad \text{and} \qquad \left[q - \tfrac{1}{n}\sqrt{2},\ \ q + \tfrac{1}{n}\sqrt{2}\right] \cap \mathbb{Q}$$

are equal, and hence these subsets are open and closed in $\mathbb{Q}$.]

(*iii*) Prove that a product of two totally disconnected topological spaces is also totally disconnnected.

5. Let U be a connected open subset of $\mathbb{R}^n$ for some positive integer n.

(*i*) Given a parametrized curve (continuous function) $\gamma : [a, b] \to U$, for some closed interval $[a, b]$, write its coordinates in the form $x_j(t)$. The curve γ is said to be a *regular piecewise smooth* curve if there is a finite sequence of points in $[a, b]$ of the form

$$a \;=\; t_0 \;<\; t_1 \;<\; \cdots \;< t_m \;=\; b$$

such that the restriction of γ to each subinterval $[t_{k-1}, t_k]$ has continuously differentiable coordinates such that the tangent vector

$$\gamma'(t) \;=\; \left(x_1'(t),\ \cdots\ , x_n'(t)\right)$$

(computed over the subinterval $[t_{k-1}, t_k]$) is always nonzero; there is no requirement that the left and right hand tangent vectors $\gamma'(t_k-)$ and $\gamma'(t_k+)$ should be the same (for example, such a parametrization exists for the boundary curve of the solid square in $\mathbb{R}^2$ defined by $0 \leq u, v \leq 1$).

Prove that if p and q are points of the open set U satisfying the conditions at the beginning of this exercise, then there is a regular piecewise smooth curve $\gamma : [a, b] \to U$ such that $\gamma(a) = p$ and $\gamma(b) = q$. [*Hint:* Define a binary relation on U such that $y \sim z$ if and only if y and z can be joined by such a curve. Show that this is an equivalence relation and that if $z \in U$ and $N_r(z; \mathbb{R}^n) \subset U$, then $N_r(z; \mathbb{R}^n)$ is contained in the equivalence class of z. From this point on, imitate the proof that an open connected subset of $\mathbb{R}^n$ is arcwise connected.]

(*ii*) Let $f : U \to \mathbb{R}$ be a continuous function such that the partial derivatives

$$\frac{\partial f}{\partial x_i} \qquad (1 \;\leq\; i \;\leq\; n)$$

are defined and equal to zero everywhere on U. Prove that f is a constant function. [*Hint:* In this case define an equivalence relation on U by $y \sim z$ if and only if $f(y) = f(z)$. Standard results on partial differentiation show that if $z \in U$ and $N_r(z; \mathbb{R}^n) \subset U$, then $N_r(z; \mathbb{R}^n)$ is contained in the equivalence class of z. From this point on, imitate the proof that an open connected subset of $\mathbb{R}^n$ is arcwise connected.]