ADDITIONAL EXERCISES FOR

MATHEMATICS 145A — Part 6

Winter 2014

13. Compact spaces

1. Let X and Y be compact Hausdorff spaces, and let $f : X \to Y$ be a function which is not assumed to be continuous. Prove that f is continuous if and only if its graph (all $(x, y) \in X \times Y$ such that $y = f(x)$) is a closed subset of $X \times Y$.

2. Suppose that X is a metric space and p is a limit point of X. Prove that there is a continuous real valued function on $X - \{p\}$ which does not take a maximum value. [*Hint:* Explain why the function $f(x) = d(x, p)$ takes values arbitrarily close to zero.]

3. (i) Let X be a metric space, let $A \subset X$ be compact, and let $B \subset X$ be a closed subset of X which is disjoint from A. If $d_B(a)$ is the distance function $d(a, B)$, prove that d_B takes a minimum value which is positive.

(ii) Give an example for $X = \mathbb{R}^2$ (with the usual Euclidean metric) such that A is a closed subset and the conclusion in (i) is not true. [*Hint:* Consider the hyperbola defined by $y = 1/x$ and one of its asymptotes.]

4. Suppose that X is a Hausdorff space and $A \subset X$ is a subspace whose closure in X is compact. Prove that the set $L(A)$ of limit points for A is also compact. [*Hint:* Why do we know that $L(A)$ is closed in X?]

14. Sequential compactness

1. A topological space X is said to be *limit point compact* if and only if every infinite subset of X has a limit point.

(i) If (X, d) is a metric space, prove that X is sequentially compact if and only if it is limit point compact.

(ii) If X is a compact topological space, prove that X is limit point compact. [*Hint:* Assume the contrary, and let S be an infinite set with no limit points. Why is every subset of S closed? Take an infinite sequence of distinct points $x_k \in S$, let $T = \{x_1, x_2,$ *etc.*$\}$, and set T_n equal to $T = \{x_n, x_{n+1},$ *etc.*$\}$. Then each T_n is nonempty and $T_n \supset T_{n+1}$ for all n, but $\cap_n T_n = \emptyset$.]

Note. The converse to (ii) is false, and two counterexamples are given on page 179 of Munkres, *Topology*. The first of these is one of the simplest to describe.

2. A family $\mathcal{E}$ of continuous real valued functions on a metric space (X, d) is said to be *equicontinuous* if for each $\varepsilon > 0$ there is some $\delta > 0$ such that $d(s, t) < \delta$ implies $|f(s) - f(t)| < \varepsilon$ for every function $f \in \mathcal{E}$. The Arzelà-Ascoli Theorem (see Goffman and Pedrick, *First Course in*

Functional Analysis, pp. 28–30, or Rudin, *Principles of Mathematical Analysis*, Third Edition, p. 158) implies that a subset $\mathcal{E}$ in the normed vector space $\mathcal{C}[0,1]$ of continuous functions from $[0,1]$ to $\mathbb{R}$ has a sequentially compact closure if and only if it is bounded and equicontinuous. — If $A, B > 0$ and $\mathcal{D}(A,B)$ is the family in the normed vector space $\mathcal{C}[0,1]$ consisting of all continuously differentiable functions f such that $|f(t)| \geq A$ and $|f'(t)| \leq B$ for all $t \in [0,1]$, show that $\mathcal{D}(A,B)$ is equicontinuous, and hence it has a sequentially compact closure by the Arzelà-Ascoli Theorem. [*Hint:* Use the Mean Value Theorem to show that $|f(s) - f(t)| \leq B \cdot |s - t|$ for all $s, t \in [0,1]$; without loss of generality, we might as well assume that $s < t$.]

Note. Other important examples of equicontinuous families are described in Section 2.15 of Goffman and Pedrick (in particular, see pages 83 – 84).