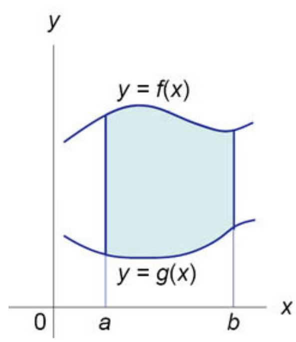
FIGURES FOR SOLUTIONS TO SELECTED EXERCISES – II

6. More concepts in metric spaces

Additional exercises

9. We begin with a drawing of a typical region being considered in this exercise:

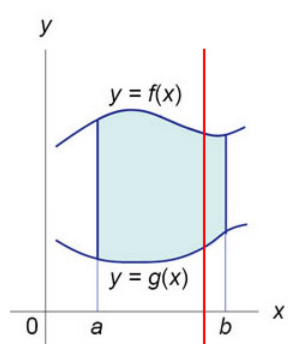


(***Source:*** **http://www.math24.net/definite-integral.html**)

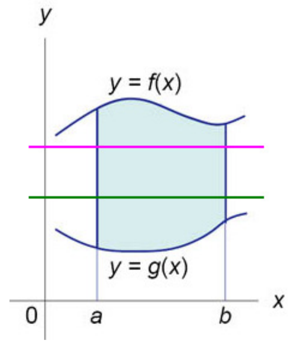
In the exercise the curves C_i are numbered in counterclockwise order starting with the bottom curve, and the corner points are the points on the graph corresponding to $x = a$ or $x = b$. Note that the two corner points at the left or at the right (or both) may coincide. The set of corner points will be denoted by E.

The main step in the solution to the exercise is to show that all the points on each of the curves C_i, with the possible exception of the points in E, are boundary points for both the open region and the closed region. Since the points in E are limit points for the curves C_i and the sets of boundary points are closed, these corner points will also be limit points for the open and closed region. Once we know these facts, the remainder of the proof is relatively straightforward.

The idea is to show explicitly that each point on a curve C_i, with the possible exception of the points in E, is a limit point of U and of the complement of A. For points on the top and bottom graphs, we find a convergent sequence of points on a vertical line through a point on the graph as in the drawing below (the vertical line is colored in red).



We also need to prove a corresponding limit statement for points on the vertical lines at the left and right of the region. In this case we want sequences which lie on horizontal lines (for example, the two colored lines in the picture below); note the second coordinates of the lines for the curves at the left and right might be different. In particular, this is unavoidable if we have a pair of functions such that $f(a) < g(b)$.



This exercise illustrates how many the concepts from Chapter 6 interact in a situation which often arises in the theory and uses of mathematics.