SOLUTIONS TO EXERCISES FOR

MATHEMATICS 145A — Part 2

Winter 2014

5. Metric spaces

Even-numbered exercises from Sutherland

5.2. Several applications of the Triangle Inequality yield

$$d(x,y) \;\leq\; d(x,z) \;+\; d(z,t) \;+\; d(t,y)\ , \qquad d(z,t) \;\leq\; d(z,x) \;+\; d(x,y) \;+\; d(y,t)$$

and if we combine these with the symmetry property for distances we obtain the inequalities

$$d(x,y) \;-\; d(z,t) \;\leq\; d(x,z) \;+\; d(y,t)\ , \qquad d(z,t) \;-\; d(x,y) \;\leq\; d(x,z) \;+\; d(y,t)\ .$$

The right hand sides of these two inequalities are identical, and each term on the left is the negative of the other. Therefore the absolute value of the left hand side is less than or equal to the right hand side, which is just the statement of the conclusion for the exercise.■

5.4. Each of the functions $f(x) = x^3$, e^x and $\tan^{-1}(x)$ is a strictly increasing real valued function, so it is enough to show that if f is such a function then $|f(x) - f(y)|$ defines a metric on $\mathbb{R}$.

Since $|f(x) - f(y)| = |f(y) - f(x)| \geq 0$, the nonnegativity and symmetry properties hold, and furthermore since $|f(x) - f(y)| = 0$ implies $f(x) = f(y)$, the strictly increasing nature of f implies that if $|f(x) - f(y)| = 0$ then $x = y$. Finally, by the properties of absolute values we have

$$|f(x) - f(z)| \;\leq\; |f(x) - f(y)| \;+\; |f(y) - f(z)|$$

so the Triangle Inequality is also satisfied. This completes the verification that $|f(x) - f(y)|$ defines a metric on $\mathbb{R}$.■

5.6. In this course we shall use $N_\eta(p)$ to denote the set called $B_\eta(p)$ in Sutherland.

We are given that $y \in N_{\varepsilon/2}(x)$, and we need to show that if $z \in N_{\varepsilon/2}(y)$ then $z \in N_\varepsilon(x)$. The given conditions imply that

$$d(x,z) \;\leq\; d(x,y) \;+\; d(y,z) \;<\; \tfrac{1}{2}\,\varepsilon \;+\; \tfrac{1}{2}\,\varepsilon \;=\; \varepsilon$$

so that $z \in N_\varepsilon(x)$, and therefore $N_{\varepsilon/2}(y)$ is contained in $N_\varepsilon(x)$.■

5.8. As usual, we have to prove implications in both directions.

$(\Leftarrow)$ By the definition of boundedness on page 50 of Sutherland, if $A \subset X$ is bounded, then there is some $x_0 \in X$ such that $d(a, x_0) \leq K$ for some $K > 0$. The Triangle Inequality then implies that for all $a, a' \in A$ we have

$$d(a,a') \;\leq\; d(a,x_0) \;+\; d(x_0,a') \;\leq\; K \;+\; K \;=\; 2K$$

so we can take $\Delta = 2K$.■

($\Rightarrow$) If $a_0 \in A$, then the hypothesis implies that $d(a_0, a) \leq \Delta$, and hence A is bounded in the sense of the previously cited definition.■

5.10. We need to show that if $x, y \in A \cup B$ then $d(x, y) \leq \operatorname{diam} A + \operatorname{diam} B$.

Since $\operatorname{diam} A, \operatorname{diam} B \leq \operatorname{diam} A + \operatorname{diam} B$, clearly the last term is an upper bounnd for $d(x, y)$ if both x and y lie in either A or B. Thus it is only necessary to verify the statement in the first paragraph when $x \in A$ and $y \in B$.

Let $p \in A \cap B$. Then $d(x, p) \leq \operatorname{diam} A$ and $d(y, p) \leq \operatorname{diam} B$ imply that

$$d(x, y) \quad \leq \quad d(x, p) \; + \; d(p, y) \quad \leq \quad \operatorname{diam} A \; + \; \operatorname{diam} B$$

so that the right hand side is an upper bound for the distances between points of $A \cup B$. Since $\operatorname{diam} A \cup B$ is the least upper bound for this set, we must have $\operatorname{diam} A \cup B \;\leq\; \operatorname{diam} A + \operatorname{diam} B$.■

5.12. *Discussion of the method of proof.* In order to show that the first three constructions define metrics, it is only necessary to verify that they satisfy all the defining properties. In order to show that the last construction does not define a metric, it suffices to show that the Triangle Inequality does not hold; in other words, for some metric space X there are points $x, y, z \in X$ such that $d^{(4)}(x, z) > d^{(4)}(x, y) + d^{(4)}(y, z)$, or equivalently $d(x, z)^2 > d(x, y)^2 + d(y, z)^2$.

Implementation of the method. To verify that $d^{(1)} = k \cdot d$ is a distance function, observe that the nonnegativity and symmetry properties follow immediately from the defining property and the fact that the product of two positive numbers is positive. Next, note that $d^{(1)}(x, y) = k \cdot d(x, y) = 0$ implies $d(x, y) = 0$, so that $x = y$. Finally, the chain of relations

$$d^{(1)}(x, z) \;=\; k \cdot d(x, y) \;\leq\; k \cdot (\, d(x, y) + d(y, z)\,) \;=\; k \cdot d(x, y) + k \cdot d(y, z) \;=\; d^{(1)}(x, y) + d^{(1)}(y, z)$$

shows that $d^{(1)}$ satisfies the Triangle Inequality.

To verify that $d^{(2)} = \min\{d, 1\}$ is a distance function, observe that the nonnegativity and symmetry properties follow immediately from the defining property and the fact that the minimum of 1 and a nonnegative number is nonnegative. Next, note that $d^{(2)}(x, y) = 0$ implies that the minimum of $d(x, y)$ and 1 is equal to zero, which means that $d(x, y) = 0$ and hence $x = y$; on the other hand, the definitions also implies that $d(x, x) = 0$. Finally, proving the Triangle Inequality requires a case by case discussion depending on whether or not the various distances are less than or equal to 1 or greater than 1.

- (a) If $d(x, z)$, $d(x, y)$ and $d(y, z)$ are all less than or equal to 1, then the Triangle Inequality for $d^{(2)}$ with this choice of x, y, z is a consequence of $d^{(2)} = d$ and the validity of the Triangle Inequality for d.
- (b) If $d(x, z) \leq 1$ and at least one of $d(x, y)$ and $d(y, z)$ is greater than 1, then $1 \leq d^{(2)}(x, y) + d^{(2)}(y, z)$ implies the Triangle Inequality for $d^{(2)}$ with this choice of x, y, z.
- (c) If $d(x, z) > 1$ and at least one of $d(x, y)$ and $d(y, z)$ is greater then $1 = d^{(2)}(x, z)$ and $1 = d^{(2)}(x, y)$ or $d^{(2)}(y, z)$ will yield yield the Triangle Inequality for $d^{(2)}$ with this choice of x, y, z.
- (d) If $d(x, z) > 1$ and both of $d(x, y)$ and $d(y, z)$ are less than or equal to 1, then $1 = d^{(2)}(x, z) < d(x, z) \leq d(x, y) + d(x, z) = d^{(2)}(x, y) + d^{(2)}(y, z)$, which yields the Triangle Inequality for $d^{(2)}$ with this choice of x, y, z.

To verify that $d^{(3)} = d/(1+d)$ is a distance function, we need some simple properties of the function which sends $d \geq 0$ to $d/(1+d)$; namely, its value is zero when $d = 0$, and the identity

$$\frac{d}{1+d} \quad = \quad 1 \; - \; \frac{1}{1+d}$$

shows that the function is strictly increasing and converges to 1 as $d \to \infty$. Once again, the nonnegativity and symmetry properties follow immediately, and if

$$0 \quad = \quad d^{(3)}(x,y) \quad = \quad \frac{d(x,y)}{1+d(x,y)}$$

then the preceding discussion implies that $d(x,y) = 0$, so that $x = y$. To verify that the Triangle Inequality holds, note that the strictly increasing nature of $d/(1+d)$ implies that

$$d^{(3)}(x,z) \quad = \quad \frac{d(x,z)}{1+d(x,z)} \quad \leq \quad \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \quad =$$

$$\frac{d(x,y)}{1+d(x,y)+d(y,z)} \; + \; \frac{d(y,z)}{1+d(x,y)+d(y,z)} \; .$$

Since $d(x,y), d(y,z) \leq d(x,y) + d(y,z)$, their reciprocals are unequal in the reverse order, which implies that the right hand side of the display is less than or equal to

$$\frac{d(x,y)}{1+d(x,y)} \; + \; \frac{d(y,z)}{1+d(y,z)} \quad = \quad d^{(3)}(x,y) \; + \; d^{(3)}(y,z).$$

This yields the Triangle Inequality for $d^{(3)}$.

Finally, to show that $d^{(4)}$ is not a metric, as noted at the beginning of this solution it is enough to find a metric space X and points $x, y, z \in X$ such that $d(x,z)^2 > d(x,y)^2 + d(y,z)^2$. One simple candidate for X is the real number system with the usual metric, and for this example we need to find x, y, z such that $(x-z)^2 > (x-y)^2 + (y-z)^2$. Since the algebra will simplify if we have as many zeros as possible, let's take $y = 0$, so that we need to find x and z such that $(x-z)^2 > x^2 + z^2$. Trial and error suggests that we take $z = -x$ and see that

$$(x-z)^2 \quad = \quad 4x^2 \; , \qquad x^2 + z^2 = 2x^2$$

so that the Triangle Inequality fails if $x = 1$, $y = 0$ and $z = -1$.∎

5.14. Suppose that $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ have coordinates x_i and y_i respectively, and let $u_i = |x_i - y_i|$. Then each u_i is nonnegative, and everything reduces to proving the following chain of inequalities:

$$\max_i u_i \quad \leq \quad \sqrt{\sum_i u_i^2} \quad \leq \quad \sum_i u_i \quad \leq \quad n \cdot \max_i u_i$$

To see the first inequality, notice that for each i we have

$$u_i^2 \quad \leq \quad \sum_j u_j^2$$

and that the inequality follows by taking square roots. Next, we have

$$\sum_i u_i^2 \quad \leq \quad \sum_i u_i^2 \;+\; 2 \cdot \sum_{j<k} u_j u_k \quad = \quad \left(\sum_i u_i \right)^2$$

and the second inequality follows from this by taking square roots. Finally, if $u_i \leq v_i$ for all i then $\sum_i u_i \leq \sum_i v_i$; if we take each v_i to be the maximum u^* of $\{u_1, \cdots, u_n\}$, then we have $\sum u_i \leq n \cdot u^*$, verifying the final inequality.■

5.16. We shall introduce some shorter notation for the various distances with arise. Specifically, if $i, j \in \{1, 2, 3\}$ let $u_{i,j} = d^X(x_i, x_j)$ and $v_{i,j} = d^Y(y_i, y_j)$. Set $\mathbf{z}_k$ equal to (x_k, y_k).

(a) The nonnegativity and symmetry properties follow immediately, so suppose that the d_p distance between $\mathbf{z}_1 = (x_1, y_1)$ and $\mathbf{z}_2 = (x_2, y_2)$ is zero, where $p = 1, 2, \infty$. This conclusion will follow if in each case we can prove that $u_{1,2} = v_{1,2} = 0$. — If $d_1 = 0$, then $u_{1,2} + v_{1,2} = 0$, and since $u_{1,2}, v_{1,2} \geq 0$ we must have $u_{1,2} = v_{1,2} = 0$. If $d_2 = 0$ then $u_{1,2}^2 + v_{1,2}^2 = 0$, which yields $u_{1,2} = v_{1,2} = 0$. Finally, if $d_\infty = 0$ then the minimum of $u_{1,2}$ and $v_{1,2}$. Since the numbers in question are nonnegative, this can only happen if $u_{1,2} = v_{1,2} = 0$.

By the preceding paragraph, we need only show that each of the d_p satsifies the Triangle Inequality. Each case must be treated separately.

$$d_1(\mathbf{z}_1, \mathbf{z}_3) \quad = \quad u_{1,3} + v_{1,3} \quad \leq \quad (u_{1,2} + u_{1,3}) \;+\; (v_{1,2} + v_{1,3}) \quad =$$

$$(u_{1,2} + v_{1,2}) \;+\; (u_{1,3} + v_{1,3}) \quad = \quad d_1(\mathbf{z}_1, \mathbf{z}_2) \;+\; d_1(\mathbf{z}_2, \mathbf{z}_3)$$

For the d_2 metric it is more convenient to compare the squares of the various expressions, using the strictly increasing nature of the square root function for nonnegative real numbers. We have

$$d_2(\mathbf{z}_1, \mathbf{z}_3)^2 \quad = \quad u_{1,3}^2 + v_{1,3}^2 \quad \leq \quad (u_{1,2} + u_{2,3})^2 \;+\; (v_{1,2} + v_{2,3})^2 \quad =$$

$$u_{1,2}^2 + u_{2,3}^2 + v_{1,2}^2 + v_{2,3}^2 + 2(u_{1,2}\, u_{2,3} + v_{1,2}\, v_{2,3})$$

and if we define vectors in $\mathbb{R}^2$ by $\alpha = (u_{1,2}, v_{1,2})$ and $\beta = (u_{2,3}, v_{2,3})$ respectively, then the right hand side is equal to $|\alpha + \beta|^2$, where $| \cdots |$ denotes the length of a vector (in the Euclidean sense). Standard results on dot products imply that $|\alpha + \beta|^2 \leq (|alpha| + |\beta|)^2$, and the Triangle Inequality follows from this and the identities

$$|\alpha| \quad = \quad \sqrt{u_{1,2}^2 + v_{1,2}^2}\;, \qquad |\beta| \quad = \quad \sqrt{u_{2,3}^2 + v_{2,3}^2}\;.$$

This proves the Triangle Inequality for the d_2 construction. — Finally, we need to verify that d_∞ satisfies the Triangle Inequality. As in the discussion for d_2, the Triangle Inequalities for the metrics on X and Y imply that

$$\max\{u_{1,3}, v_{1,3}\} \quad \leq \quad \max\{u_{1,2} + u_{2,3}, v_{1,2} + v_{1,3}$$

and the right hand side is less than or equal to $\max\{u_{1,2}, v_{1,2}\} + \max\{u_{2,3}, v_{2,3}\}$. The latter proves the Triangle Inequality for the d_∞ construction.■

Note. In fact, there is a continuous family of product metrics d_p on $X \times Y$, where $1 \leq p < \infty$ defined by the formula

$$d_p(\mathbf{z}_1, \mathbf{z}_2) \quad = \quad (\, d(x_1, x_2)^p \;+\; d(u_1, y_2)^p\,)^{1/p}$$

(with the same notation as in part (b) of the exercise) which generalizes the metrics d_1 and d_2 and also satisfies the conditions

$$p \quad < \quad q \qquad \Rightarrow \qquad d_q \quad \leq \quad d_p \ , \qquad\qquad \lim_{p\to\infty} \ d_p \quad = \quad d_\infty \ .$$

There are further details regarding these constructions and proofs of the assertions about them in the files `product-metrics.pdf`, `product-metrics2.pdf` and `product-metrics3.pdf`.

5.18. There are plenty of counterexamples for which both $x \neq y$ and $r \neq s$. In particular, if X is a bounded metric space with diameter Δ and X contains two points x and y, then $N_\Delta(x) = N_{2\Delta}(y)$. Of course, the simplest example is $\{x, y\}$ where $d(x, y) = 1$, in which case $\Delta = 1$.■

Additional exercise(s)

1. Since $N_{\delta(x)} \subset U$ for all x we have $\cup_{x\in U} \ N_{\delta(x)} \subset U$. To prove the reverse inclusion note that

$$U \quad = \quad \bigcup_{x\in U} \{x\} \quad \subset \quad \bigcup_{x\in U} N_{\delta(x)} \quad \subset \quad U \ .■$$

2. Since X is finite the set D of all real numbers of the form $d(u, v)$, where u and v run through all distinct pairs of points in X, is also finite, and therefore D has a positive minimum element m. Therefore, if $x \in X$ then the open set $N_{m/2}(x)$ only contains the point x itself, and consequently every one point subset of X is open. Since every subset in X is a union of one point subsets and a union of open sets is open, it follows that every subset of X is open.■

3. If f is an open mapping with U open in X and $x \in U$, we can take $W_x = U$. Conversely, suppose that the condition in the exercise is satisfied; we need to prove that if U is open in X, then $f[U]$ is open in Y. Given $U \subset X$ open and $x \in X$, let some open set W_x in X such that $W_x \subset U$ be an open ubset with $x \in X$ and such that $f[W_x]$ is open in Y. As in the preceding exercise we have $\cup_{x\in U} \ W_x \ = \ U$, and therefore

$$f[U] \quad = \quad \bigcup_{x\in U} \ f[W_x]$$

and since the sets $f[W_x]$ are all open in Y it follows that $f[U]$ is also open in Y.■

4. By Proposition 3.20 in Sutherland, we have $f[V] = h^{-1}[V]$, and since f is the inverse function to h we also have $h[W] = f^{-1}[W]$.

If h is open, then the identity in the preceding sentence implies that $h[W]$ is open in X if W is open in Y, and therefore f is continuous. Conversely, if f is continuous, then $h[W] = f^{-1}[W]$ implies that $h[W]$ is open in X if W is open in Y, and therefore h is an open mapping.■

5. Suppose first that we have a space with the standard discrete metric. If $x = y$, then $d(x, y) = 0 \leq \max\{d(x, z),\ d(y, z)\}$ because the two numbers on the right hand side are nonnegative; on the other hand, if $x \neq y$, then $d(x, y) = 1$ and either $x \neq z$ or $y \neq z$. No matter which of the latter holds we have $\max\{d(x, z),\ d(y, z)\} = 1$, and hence the defining condition for an ultra-metric space is satisfied.■

On the other hand, the real line is **NOT** an ultra-metric space. One easy counterexample is given by $x = 0$, $z = \frac{1}{2}$ and $y = 1$, in which case $d(x, y) = 1$ but $\max\{d(x, z),\ d(y, z)\} = \frac{1}{2}$.■

NOTE. Example 5.11 on page 44 of Sutherland is a less trivial example of an ultra-metric space, and in fact the verification of the ultrametric inequality $d(m,q) \leq \max\{d(m,n),\ d(n,q)\}$ is a key step in the derivation of the Triangle Inequality for this example.■

6. (*i*) Let $c = \frac{1}{2}$ and set A_1 and A_2 be the closed intervals $[0,c]$ and $[c,1]$ respectively. If we set

$$g_1(t) \;=\; \tfrac{1}{2}\,t\ , \qquad g_2(t) \;=\; \tfrac{1}{2}\,t \;+\; \tfrac{1}{2}$$

then these maps satisfy the condition in the definition with $r = \frac{1}{2}$.■

(*ii*) Take A_1 to be the piece of the Cantor set which lies in the interval $[0, \frac{1}{3}]$, and take A_2 to be the piece of the Cantor set which lies in the interval $[\frac{2}{3}, 1]$. The maps g_1 and $g-2$ are the linear functions

$$g_1(t) \;=\; \tfrac{1}{3}\,t\ , \qquad g_2(t) \;=\; \tfrac{1}{3}\,t \;+\; \tfrac{2}{3}$$

then these maps satisfy the condition in the definition with $r = \frac{1}{3}$. To be completely rigorous, it would be necessary to verify that each of these is 1–1 onto the appropriate piece of the Cantor set, but the problem does not require that this step be completed (it turns out to be straightforward but somewhat messy).■

(*iii*) The hypotheses imply that A is a union of finitely many subsets $A_1, \ \cdots\ , A_m$ such that for each $k = 1, ..., m$ there is a 1–1 correspondence $g_k : A \to A_k$ which multiplies distances by a factor of r, and furthermore B is a union of finitely many subsets $B_1, \ \cdots\ , B_q$ such that for each $k = 1, ..., m$ there is a 1–1 correspondence $h_j : B \to B_j$ which multiplies distances by a factor of r. It follows that $C = A \times B$ is a union of the sets $A_k \times B_j$; furthermore, if $\varphi_{k,j}$ is the map $g_k \times h_j$, we need to verify that $\varphi_{k,j}$ multiplies distances by a factor of r. — This will follow from the chain of equations

$$d_2\big(\,g_k \times h_j(a,b),\ f_k \times g_j(a',b')\,\big) \;=\; \Big(d\big(\,(g_k(a), g_k(a')\,\big)^2 \;+\; d\big(\,(h_j(b), h_j(b')\,\big)^2\Big)^{1/2} \;=$$

$$\big(r^2 d(a,a')^2 + r^2 d(b,b')^2\big)^{1/2} \;=\; r \cdot d_2\big(\,(a,b),\ (a',b')\,\big)\ .$$

Note that the number of pieces is mq.■

(*iv*) Assume that A is bounded and let D be the set of distances between points of A. Since f is 1–1 onto and multiplies distances by a factor of r, it follows that $r \cdot D$ is also the set of all distances for D, so that $D = r \cdot D$. If the least upper bound of D is Δ, then the least upper bound for $r \cdot D$ is $r \cdot \Delta$, and therefore we have $\Delta = r \cdot \Delta$. Since A contains more than one point, it follows that $\Delta > 0$ and therefore $r \cdot \Delta = \Delta$ implies $r = 1$.■

(*v*) Take $A = [0,1]$ and $f(t) = 1 - t$. Then f is equal to its own inverse function, and $|f(u) - f(v)| \;=\; |v - u|$ by construction, so f multiplies distances by a factor of 1.■

7. We need to verify that F is continuous at every point $x \in \mathbb{R}$. Suppose first that $x \in [a,b]$. If $\varepsilon > 0$, let δ be such that $t \in [a,b]$ and $|t - x| < \delta$ implies $d(f(t), f(x)) < \varepsilon$. If $(x - \delta, x + \delta)$ is completely contained in $[a,b]$, then it follows that F is also continuous at x. Suppose now that $(x - \delta, x + \delta)$ is NOT completely contained in $[a,b]$, so that either $a \in (x - \delta, x]$ or $b \in [x, x + \delta)$; it is possible that both of the latter are true. In any case, if $t \leq a$ and $t, a \in (x - \delta, x]$ then $F(t) = f(a)$ and hence $d(F(t), F(x)) = d(f(a), f(x)) < \varepsilon$, while if $t \geq b$ and $t, b \in [x, x + \delta)$ then we have $d(F(t), F(x)) = d(f(b), f(x)) < \varepsilon$. This shows that F is continuous at $x \in [a,b]$ even if $(x - \delta, x + \delta)$ is not completely contained in $[a,b]$.

Suppose now that $x \notin [a,b]$ so that $x < a$ or $x > b$; if $x = a$ take $\delta = a - x$, while if $x > b$ take $\delta = x - b$. Then $|t - x| < \delta$ implies that $(x - \delta, x + \delta)$ is contained in the complement of $[a,b]$ and

F is constant on $(x-\delta, x+\delta)$, with $F(x) = f(a)$ if $x < a$ and $F(x) = f(b)$ if $x > b$. Therefore in these cases we also know that $|t-x| < \delta$ implies $d(F(t), F(x)) = 0 < \varepsilon$.∎

6. More concepts in metric spaces

Even-numbered exercises from Sutherland

6.2. (a) is closed because its complement is $(-\infty, 1)$, which is open.∎

(b) is not closed; we shall verify that is complement $\mathbb{Q}$ is not open. If $x \in \mathbb{Q}$ and $\varepsilon > 0$, then $N_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$ contains irrational numbers by a previous exercise, so no such neighborhood is entirely contained in $\mathbb{Q}$.∎

(c) is not closed because

$$\lim_{n\to\infty} \frac{n}{n+1} \quad = \quad 1$$

and 1 does not belong to the subset, which means that the subset does not contain all its limit points and therefore is not closed.∎

(d) is closed, for its complement is the union of the open intervals

$$(-\infty, 0)\ , \quad (2, \infty)\ , \quad (1, 2)\ , \quad \text{and} \quad \left(\frac{1}{n+1}, \frac{1}{n}\right)$$

where in the last case n runs through all the positive integers.∎

6.4. Suppose that F_α is closed in X for $\alpha \in A$; we want to show that the complement of the intersecton is open. But the latter is equal to

$$X \;-\; \left(\bigcap_{\alpha\in A} F_\alpha\right) \quad = \quad \bigcup_{\alpha\in A} X - F_\alpha$$

and all of the sets in the expression on the right are open because the subsets F_α are all closed. Therefore the complement of the intersection is a union of open sets and hence is open, so the intersection itself must be closed.∎

6.6. One crucial point is to describe the complement correctly. It is the set of all functions f such that $f(a) \neq 0$ for at least some choice(s) of a in A. So assume f lies in the complement of the set in question, and choose $a_0 \in A$ such that $f(a_0) \neq 0$. If g is a continuous function such that $|g - f| < \frac{1}{2}|f(a_0)|$, then $g(a_0)$ is also nonzero and hence g lies in the complementary subset, which means that the latter must be open in the space of continuous functions with the uniform metric.∎

6.8. By Additional Exercise 6.1 the set of all $y \in \mathbb{R}^2$ satisfying $|y| \leq 1$ is closed in $\mathbb{R}^2$, so the closure of $N_1(0)$ is contained in this subset. We need to verify that every point y satisfying $|y| = 1$ is a limit point of $N_1(0)$. But if we let $c_n = 1 - \frac{1}{n}$ where n runs over all integers ≥ 2, then it follows that $c_n \cdot y \in N_1(0)$ and $c_n \cdot y \longrightarrow y$ as $n \to \infty$.∎

6.10. Suppose that A is bounded and has diameter Δ. If x and y lie in the closure of A, then there are sequences $\{a_n\}$ and $\{b_n\}$ in A whose limits are x and y respectively (we have shown this for limit points of A; if $a \in A$ is not a limit point, take the sequence whose terms are all equal to A).

CLAIM: $d(x,y) = \lim_{n\to\infty} d(a_n,b_n)$. — Assuming this is true, we shall prove the assertions in the exercise. The main thing to verify is that $d(x,y) \leq \Delta$; if this were not the case and $h = d(x,y) - \Delta > 0$, then for some N we would have $|d(x_n,y_n) - d(x,y)| < h$ for $n \geq N$, and the latter would imply $d(x_n,y_n) > d(x,y) - h = \Delta$, contradicting the assumption that Δ is the diameter of A. Hence we do have $d(x,y) \leq \Delta$. By the definition of diameter this implies that the diameter of $\overline{A}$ is at most Δ. On the other hand, since $A \subset \overline{A}$ we also know that $\Delta \leq \operatorname{diam} \overline{A}$, and if we combine this with the previous observation we see that $\Delta = \operatorname{diam} \overline{A}$.

Proof of the limit formula. Given $\varepsilon > 0$ we can choose N so large that $d(x,a_n) < \frac{1}{2}\varepsilon$ and $d(y,b_n) < \frac{1}{2}\varepsilon$ provided $n \geq N$. Therefore Exercise 5.2 in Sutherland implies that

$$|d(x,y) \;-\; d(a_n,b_n)| \;\;\leq\;\; d(x,a_n) \;+\; d(y,b_n) \;\;<\;\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;\;=\;\; \varepsilon$$

and this proves the limit formula.■

6.12. Let $A_r(x)$ denote the set of all points $y \in X$ such that $d(x,y) \leq r$. Then Additional Exercise 6.1 (see below) shows that $A_r(x)$ is closed, and therefore we have

$$\overline{N_r(x)} \quad \subset \quad A_r(x)$$

because the closure of a subset S is the smallest closed subset containing S. Following the hint, we shall construct a counterexample involving discrete metric spaces. Specifically, suppose that X is discrete and has more than one element. Then $A_1(x) = X$ strictly contains $N_1(x) = \{x\}$, and the latter is closed because X is discrete. Therefore the closure of $N_1(x)$ is strictly contained in $A_1(x)$ for this example.■

6.14. If we can prove the result when $m = 2$, then we can also prove it for all finite values of m by an argument using mathematical induction, so we shall concentrate on the special case $m = 2$.

Since the closure construction **Clos** satisfies $A \subset B \Rightarrow \mathbf{Clos}\,(A) \subset \mathbf{Clos}\,(B)$, it follows that $\mathbf{Clos}\,(A_1)$ and $\mathbf{Clos}\,(A_2)$ are contained in $\mathbf{Clos}\,(A_1 \cup A_2)$, it follows that

$$\mathbf{Clos}\,(A_1) \;\cup\; \mathbf{Clos}\,(A_2) \quad\subset\quad \mathbf{Clos}\,(A_1 \cup A_2)\;.$$

Conversely, since $\mathbf{Clos}\,(A_1) \cup \mathbf{Clos}\,(A_2)$ is closed, it follows that

$$\mathbf{Clos}\,(A_1 \cup A_2) \quad\subset\quad \mathbf{Clos}\,(A_1) \;\cup\; \mathbf{Clos}\,(A_2)$$

and the conclusion of the exercise follows from the two displayed inclusions.■

6.16. (a) If $x \in \overline{A}$, then x belongs to either A or $L(A)$. In the first case $0 = d(x,x)$ implies that $d(x,A) = 0$, and in the second case there is a sequence $\{a_n\}$ in A such that $\lim_{n\to\infty} a_n = x$. By the continuity of the distance function it follows that $0 = \lim_{n\to\infty} d(a_n,x)$, which means that $0 = d(x,A)$. Conversely, suppose that $d(x,A) = 0$ but $x \notin A$. Then for each positive integer n there is some $a_n \in A$ such that $d(a_n,x) < \frac{1}{n}$, which means that $x \in L(A)$.■

(b) If $x,y \in X$ and $a \in A$, then $d(x,a) \leq d(x,y) + d(y,a)$. Since $d(x,A)$ is the greatest lower bound of the numbers $d(x,a)$, it follows that $d(x,A) \leq d(x,y) + d(y,a)$. Therefore we also have $d(x,A) - d(x,y) \leq d(y,a)$ for all $a \in A$, and since $d(y,A)$ is the greatest lower bound of the numbers $d(y,a)$, it follows that $d(x,A) - d(x,y) \leq d(y,A)$.

(c) If we interchange the roles of x and y in the previous sentence, we obtain a second inequality $d(y, A) - d(x, y) \leq d(x, A)$. We can write this and the preceding inequality in the combined form $|d(x, A) - d(y, A)| \leq d(x, y)$ and therefore $d(x, y) < \varepsilon$ implies that $|d(x, A) - d(y, A)| < \varepsilon$, so that $h(x) = d(x, A)$ is uniformly continuous.■

6.18. Let S be the finite subset of the metric space. We have seen in the lectures that every for one point subset of S is equal to $N_\delta(x) \cap S$ for some $\delta > 0$ depending on x; it follows that $\{x\} = N_\delta(x) \cap S$, so that $(N_\delta(x) - \{x\}) \cap S = \emptyset$. This means that x is not a limit point of S.■

6.20. If f is continuous, then $f^{-1}[\mathrm{Int}\, B]$ is open and contained in $f^{-1}[B]$. so we have

$$f^{-1}[\mathrm{Int}\, B] \quad \subset \quad \mathrm{Int}\, f^{-1}[B] \; .$$

Conversely, suppose that the latter holds for every subset B. In particular, if U is an open set, then we have

$$f^{-1}[U] \quad \subset \quad \mathrm{Int}\, f^{-1}[U] \quad f^{-1}[U]$$

which means that $f^{-1}[U]$ is equal to its own interior and hence is open in X■

6.22. If $x \in \mathrm{Bdy}\,(A)$, then every open neighborhood of x contains points of A and $X - A$, so that $d(x, A)$ and $d(x, X - A)$ are both less than or equal to ε for each $\varepsilon > 0$. Therefore $d(x, A) = d(x, X - A) = 0$. Conversely, if the latter holds then for every $\varepsilon > 0$ then there are points in $N_\varepsilon(x) \cap A$ and $N_\varepsilon(x) \cap (X - A)$. By Proposition 6.24 this implies that $x \in \mathrm{Bdy}\,(A)$.■

6.24. Let L be the limit of the sequence. Given $\varepsilon > 0$, choose N so that $m, n \geq N$ imply $d(a_p, L) < \frac{1}{2}\,\varepsilon$ for $p = m, n$. By the Triangle Inequality we have $d(a_m, a_n) < \varepsilon$, which means that the convergent sequence is a Cauchy sequence.■

6.26. If a is a limit point of Y, then this result is true by the lemma on page 6.10 of the file `math145Anotes06.pdf`. On the other hand, if $a \in Y$, then we can take the sequence such that $a_n = a$ for all n.■

Additional exercise(s)

1. (i) Following the hint, we want to show that if $z \in N_\varepsilon(y)$, where $d(y, x) > r$ and $\varepsilon = d(x, y) - r$, then $d(z, x) > r$. This follows because $d(z, x) \geq d(y, x) - d(z, y)$, $d(y, x) > r$ and $d(z, y) < d(x, y) - r$ imply that

$$d(z, x) \quad > \quad d(x, y) \; - \; \big(d(x, y) - r\big) \quad = \quad r \; .$$

(ii) The complement of the set in question is the set $\{y \mid d(x, y) > r\}$, and the conclusion follows because the latter is open and the complement of an open set is closed.■

2. Let $L(A; B)$ be the set of all limit points for A which lie in B. Then the characterization of limit points in terms of limits of sequences in A (such that no term in the sequence equals the limit), it follows that $L(A) \cap B = L(A; B)$. Therefore the closure of A in B is equal to $A \cup L(A; B) = A \cup (L(A, B) \cap B)$. Since $A \subset B$, it follows that

$$\mathbf{Clos}\,(A; B) \;=\; A \cup L(A; B) \;=\; (A \cap B) \cup (L(A, B) \cap B) \;=\; (A \cup L(A)) \cap B \;=\; \mathbf{Clos}\,(A) \cap B$$

which is what we wanted to prove.■

3. For every $x \in X$ we have

$$\{x\} \quad = \quad \bigcap_n N_{1/n}(x)$$

so if an intersection of countable open subsets is always open, then every one point subset is open. Since the union of open sets is open, it follows that every subset is open (and also every subset is closed).■

4. The map sending x to (x, b) is 1–1 and onto, and an explicit inverse function is given by sending (x, b) to x. For $p = 1, 2, \infty$ the definitions of the product metrics imply that

$$d^X(x, x') \quad = \quad d_p\big(\, (x, b), \, (x', b) \, \big)$$

so that the map in the first sentence of this paragraph is an isometry.

The proof of the companion result follows by a few substitutions of variables in the preceding argument: The map sending y to (a, y) is 1–1 and onto, and an explicit inverse function is given by sending (a, y) to y. For $p = 1, 2, \infty$ the definitions of the product metrics imply that

$$d^Y(y, y') \quad = \quad d_p\big(\, (a, y), \, (a, y') \, \big)$$

so that the map in the first sentence of this paragraph is an isometry.■

5. (i) Suppose that $X - \{p\}$ is dense in X. Then x must be a limit point of X, and hence for each $\varepsilon > 0$ the open set $N_\varepsilon(x)$ contains points of $X - \{p\}$, and therefore $\{p\}$ is not open in X. Conversely, if $X - \{p\}$ is not dense in X, then the closure of $X - \{p\}$, which is contained in $X = (X - \{p\}) \cup \{p\}$, must be $X - \{p\}$ itself, and consequently this subset is closed. Therefore its complement, which is $\{p\}$, must be open in X.■

(ii) Since the closure of a subset is the union of the latter with its limit points, if $D \subset X$ and $x \in X$ then for every $\varepsilon > 0$ there is some point $y \in N_\varepsilon(x) \cap D$. — If we apply this first to the dense subset U, then this yields a point $y \in N_\varepsilon(x) \cap U$. Now choose $\delta > 0$ such that $N_\delta(y) \subset N_\varepsilon(x) \cap U$. Since V is dense, it follows that there is some $z \in N_\delta(y) \cap V \subset N_\varepsilon(x) \cap (U \cap V)$. But this implies that $U \cap V$ is dense in X.■

(iii) Let $X = \mathbb{Q}$ with the usual metric, and for each $q \in \mathbb{Q}$ let $V(q) - \mathbb{Q} - \{q\}$. The family of subsets $\{V(q)\}$ is countable because the rationals are countable, each of these subsets open, and each is dense because $N_\varepsilon(q)$ is nonempty, but the intersection of the family $V(q)$ is empty and therefore not dense in $\mathbb{Q}$.■

6. We shall first prove that the interior of H in $\mathbb{R}^n$ is empty. Suppose to the contrary that there is some $p \in H$ and some $\varepsilon > 0$ such that $N_\varepsilon(p) \subset H$. Then by the definition of H we know that $F = 0$ on $N_\varepsilon(p)$. However, we have

$$F(p + ta) \quad = \quad F(p) \; + \; t|a|^2 \quad = \quad t|a|^2$$

so $p + ta \notin H$ for $t \neq 0$; since $t < \varepsilon/|a|$ implies that $p + ta \in N_\varepsilon(p)$, we cannot have $N_\varepsilon(p) \subset H$. This contradiction implies that the interior of H is empty. Note also that H is closed because it is the zero set of a continuous real valued function.

We shall now prove that $\mathbb{R}^n - H$ is dense in $\mathbb{R}^n$. If this were not the case, then the complement of the closure contains some open subset of the form $N_r(q)$, and this open subset would have to be contained in H. Since H has an empty interior, this cannot happen, and therefore $\mathbb{R}^n - H$ must be dense in $\mathbb{R}^n$.■

7. (i) By the cited exercise in Sutherland the map $d(x, A)$ is a continuous function of x. Therefore, for each positive integer n the set W_n of all x such that $d(x, A) < \frac{1}{n}$ is open. If F is closed in X, then we know that $x \in F$ if and only if $d(x, F) = 0$, and therefore we have

$$F \quad = \quad \bigcap_n W_n$$

so that F is a countable intersection of open subsets.■

(ii) Given the open set U let $F = X - U$, which is closed in X. By (i) we know that $F = \cup_n W_n$ where each W_n is open in X. Therefore we have

$$U \quad = \quad X \ - \ F \quad = \quad X \ - \ \left(\bigcap_n W_n\right) \quad = \quad \bigcup_n X \ - \ W_n$$

where each of the sets $X - W_n$ is closed in X.■

8. If $x \in \mathrm{Bdy}\,(C \cap Y, Y)$, then for each $\varepsilon > 0$ the set $N_\varepsilon(x; Y)$ contains points of $C \cap Y$ and $Y - (C \cap Y)$ by Proposition 6.24 in Sutherland. Since $C \cap Y \subset C$ and $Y - (C \cap Y) \subset X - C$, then the same proposition implies that $x \in \mathrm{Bdy}\,(C, X)$.

To see that containment may be proper, let $C = [0, 1] \subset x = \mathbb{R}$ and let $Y = [0, 2] \subset X = \mathbb{R}$. Then $\mathrm{Bdy}\,(C \cap Y, Y) = \{1\}$ but $\mathrm{Bdy}\,(C, X) = \{0, 1\}$.■

Comment. When looking for counterexamples like the preceding one, it is usually extremely worthwhile to draw some pictures and use them to find candidates for counterexamples.

9. We shall refer to `solutions02w14.figures.pdf` for motivation; the curves C_i, where $1 \leq i \leq 4$, are defined in the statement of the problem.

Let E be the corner points of A:

$$E \quad = \quad \{(a, g(a)) \ , \ (a, f(a)) \ , \ (b, g(b)) \ , \ (b, f(b))\}$$

CLAIM: For $1 \leq i \leq 4$ every point of $C_i - E$ is a limit point for each of the sets A, $\mathbb{R}^2 - A$, V and $\mathbb{R}^2 - V$.

The set $C_1 - E$. — Suppose that $a < x < b$, so that $(x, g(x)) \in C_1 - E$, and consider the sequences $\{p_n\}$ and $\{q_n\}$ defined by

$$p_n \quad = \quad \big(x, g(x) - \tfrac{1}{n}\big) \qquad \text{and} \qquad q_n \quad = \quad \big(x, g(x) + \tfrac{1}{n}\big)$$

respectively. Then we have

$$\lim_{n \to \infty} \ p_n \quad = \quad (x, g(x)) \quad = \quad \lim_{n \to \infty} \ q_n$$

and $q_n \in \mathbb{R}^2 - A \subset \mathbb{R}^2 - V$ for all n, so that $(x, g(x))$ is a limit point of both $\mathbb{R}^2 - A$ and $\mathbb{R}^2 - V$. To show that $(x, g(x))$ is a limit point of both A and V, it will suffice to show that $p_n \in V \subset A$ if n is sufficiently large. The latter follows because $p_n < f(x)$ if $\frac{1}{n} < f(x) - g(x)$.

The set $C_3 - E$. — Suppose that $a < x < b$, so that $(x, f(x)) \in C_3 - E$, and consider the sequences $\{p_n\}$ and $\{q_n\}$ defined by

$$p_n \quad = \quad \left(x, f(x) - \tfrac{1}{n}\right) \qquad \text{and} \qquad q_n \quad = \quad \left(x, f(x) + \tfrac{1}{n}\right)$$

respectively. Then we have

$$\lim_{n\to\infty} p_n \quad = \quad (x, f(x)) \quad = \quad \lim_{n\to\infty} q_n$$

and $q_n \in \mathbb{R}^2 - A \subset \mathbb{R}^2 - V$ for all n, so that $(x, f(x))$ is a limit point of both $\mathbb{R}^2 - A$ and $\mathbb{R}^2 - V$. To show that $(x, f(x))$ is a limit point of both A and V, it will suffice to show that $p_n \in V \subset A$ if n is sufficiently large. The latter follows because $p_n > g(x)$ if $\frac{1}{n} < f(x) - g(x)$.

The set $C_2 - E$. — This set is empty if $g(b) = f(b)$, so suppose that $g(b) < f(b)$ and $g(b) < y < f(b)$, which means that $(b, y) \in C_2 - E$. Consider the sequences $\{p_n\}$ and $\{q_n\}$ defined by

$$p_n \quad = \quad \left(b - \tfrac{1}{n}, y\right) \qquad \text{and} \qquad q_n \quad = \quad \left(b + \tfrac{1}{n}, y\right)$$

respectively. Then we have

$$\lim_{n\to\infty} p_n \quad = \quad (b, y) \quad = \quad \lim_{n\to\infty} q_n$$

and $q_n \in \mathbb{R}^2 - A \subset \mathbb{R}^2 - V$ for all n, so that (b, y) is a limit point of both $\mathbb{R}^2 - A$ and $\mathbb{R}^2 - V$. To show that (b, y) is a limit point of both A and V, it will suffice to show that $p_n \in V$ if n is sufficiently large. To prove this, note that by the continuity of f and g we can find some $\delta > 0$ such that $b - x < \delta$ implies

$$|f(x) - f(b)| \quad < \quad f(b) - y \qquad \text{and} \qquad |g(x) - g(b)| \quad < \quad y - g(b)\ .$$

Therefore if we choose N such that $1/N < \delta$, then

$$g\left(b - \tfrac{1}{n}\right) \quad y \quad f\left(b - \tfrac{1}{n}\right)$$

so that $(b, y) \in V \subset A$.

The set $C_4 - E$. — This set is empty if $g(a) = f(a)$, so suppose that $g(a) < f(a)$ and $g(a) < y < f(a)$, which means that $(a, y) \in C_4 - E$. Consider the sequences $\{p_n\}$ and $\{q_n\}$ defined by

$$p_n \quad = \quad \left(a - \tfrac{1}{n}, y\right) \qquad \text{and} \qquad q_n \quad = \quad \left(a + \tfrac{1}{n}, y\right)$$

respectively. Then we have

$$\lim_{n\to\infty} p_n \quad = \quad (a, y) \quad = \quad \lim_{n\to\infty} q_n$$

and $p_n \in \mathbb{R}^2 - A \subset \mathbb{R}^2 - V$ for all n, so that (a, y) is a limit point of both $\mathbb{R}^2 - A$ and $\mathbb{R}^2 - V$. To show that (a, y) is a limit point of both A and V, it will suffice to show that $p_n \in V$ if n is sufficiently large. To prove this, note that by the continuity of f and g we can find some $\delta > 0$ such that $x - a < \delta$ implies

$$|f(x) - f(a)| \quad < \quad f(a) - y \qquad \text{and} \qquad |g(a) - g(b)| \quad < \quad y - g(a)\ .$$

Therefore if we choose N such that $1/N < \delta$, then

$$g\left(a + \tfrac{1}{n}\right) \quad < \quad y \quad < \quad f\left(a + \tfrac{1}{n}\right)$$

so that $(a, y) \in V \subset A$.

This completes the verification of the claim, which is the main step in our solution for this exercise.

To simplify notation write $C = C_1 \cup C_2 \cup C_3 \cup C_4$. By the preceding claim we know that $C - E$ is contained in the sets $L(A) \cap L(\mathbb{R}^2 - A)$ and $L(V) \cap L(\mathbb{R}^2 - V)$, and hence these sets are contained about $\mathrm{Bdy}\,(A)$ and $\mathrm{Bdy}\,(V)$ respectively.

We want to show that all of C is contained in these sets; in other words, we want to show that points of E are also limit points of A, $\mathbb{R}^2 - A$, V and $\mathbb{R}^2 - V$. By definition, the set of boundary points for a subset S is $\overline{S} - \mathrm{Int}\,(S)$, so it is an intersection of two closed sets and accordingly is closed in X. Let n be so large that that

$$a \quad < \quad a + \tfrac{1}{n} \quad < \quad b - \tfrac{1}{n} \quad < \quad b \; .$$

Then for n sufficiently large we know that $C - E$ contains the sequences with terms

$$\left(a + \tfrac{1}{n}, g(a + \tfrac{1}{n})\right) \ , \qquad \left(a + \tfrac{1}{n}, f(a + \tfrac{1}{n})\right) \ , \qquad \left(b - \tfrac{1}{n}, g(b - \tfrac{1}{n})\right) \ , \qquad \left(b - \tfrac{1}{n}, f(b - \tfrac{1}{n})\right)$$

and by the continuity of f and g the limit of these sequences are equal to

$$(a, g(a)) \ , \ (a, f(a)) \ , \ (b, g(b)) \ , \text{ and } (b, f(b))$$

respectively, so that E is contained in the set of limit points of $C - E$. Since sets of boundary points are closed it follows that E is contained in the sets $L(A) \cap L(\mathbb{R}^2 - A)$ and $L(V) \cap L(\mathbb{R}^2 - V)$. Therefore $C = (C - E) \cup E$ is contained in these two sets, so that C is contained in $\mathrm{Bdy}\,(A)$ and $\mathrm{Bdy}\,(V)$.

To complete the argument we need to check that (a) the boundaries do not contain anything else, (b) the closure of U is A, (c) the interior of A is U.

By construction we have $A = V \cup C$, where A is closed, V is open, $C \cap V = \emptyset$, and $C \subset L(A) \cap L(V)$. The first and last parts of the preceding sentence imply that $A \subset \overline{V}$; since A is closed, we also have $\overline{V} \subset A$, and therefore these two sets are equal. To prove (b), note that $V \subset A$ implies that $V \subset \mathrm{Int}\,(A)$; on the other hand, the claim implies that $C \cap \mathrm{Int}\,(A) = \emptyset$, and therefore we must have $V = \mathrm{Int}\,(A)$. We can now apply the preceding identity $\mathrm{Bdy}\,(S) = \overline{S} - \mathrm{Int}\,(S)$ to conclude that $\mathrm{Bdy}\,(A) = \mathrm{Bdy}\,(V) = C$.■