SOLUTIONS TO EXERCISES FOR

MATHEMATICS 145A — Part 3

Winter 2014

7. Topological spaces

Even-numbered exercises from Sutherland

7.2. The simplest example involves the Sierpiński space topology: Let $X = \{0.1\}$, let $\mathcal{T}_0$ consist of $\{\emptyset, \{0\}, X\}$, and let $\mathcal{T}_1$ consist of $\{\emptyset, \{1\}, X\}$. By Example 7.7 these are topologies on X, and neither one is contained in the other. — One can also give many examples of metrics on the same set such that their associated topologies have this property, but at this point we do not have the machinery needed to prove anything.■

7.4. Let U_n be the set $\{1, \cdots, n\}$. Then $U_p \cap U_q = U_r$ where r is the smaller of p and q. Suppose now that U_α belongs to the family, and consider the union

$$\bigcup_\alpha U_\alpha \ .$$

If some $U_\alpha = \mathbb{N}$, then the union is equal to $\mathbb{N}$ and hence the union is open. Suppose now that each U_α is finite, and let $n(\alpha)$ denote its maximal element. If there are only finitely many values $n(\alpha)$, then the union is U_m where m is the maximal value $n(\alpha)$. On the other hand, if there is no maximal value, then the union is equal to $\mathbb{N}$. In each of this case a union of sets in the family also belongs to the family, and therefore the family forms a topology on $\mathbb{N}$.■

7.6. The intersection of two sets $(-\infty, b_1)$ and $(-\infty, b_2)$ is equal to $(-\infty, b^*)$, where b^* is the smaller of b_1 and b_2. As in the preceding exercise, it suffices to show that a nonempty union of proper subsets from the family is also a member of that family. Suppose then that we have proper subsets $(-\infty, b_\alpha)$ from this family. Then the union

$$\bigcap_\alpha (-\infty, b_\alpha)$$

is equal to $(-\infty, c)$ where $c = +\infty$ if the set $B = \{b_\alpha\}$ has no upper bound, and c is the least upper bound of B if the latter has an upper bound. In each case the union of sets from the family still belongs to this family, so the family does form a topology on $\mathbb{R}$.■

IMPORTANT NOTE. Henceforth, the solutions to ALL exercises in Sutherland with be in files that have been distributed by the book author and will simply be appended to the course directory files.

Additional exercise(s)

1. (*i*) Suppose that f is upper semi-continuous, let $x \in X$, and let $\varepsilon > 0$. Then the inverse image of $(-\infty, f(x) + \varepsilon)$ is open, and hence it contains some open subset U such that $x \in U$.

Conversely, suppose that the condition in the exercise holds, let $(-\infty, b) \in \mathcal{U}$, and let x be a point in the inverse image D of $(-\infty, b)$. If $\varepsilon = b - f(x)$, then $\varepsilon > 0$ and hence there is some open set V_x in X such that $x \in V_x$ and f maps V_x into $(-\infty, b)$. Therefore the inverse image is equal to

$$\bigcup_{f(x)\in D} \{x\} \quad = \quad \bigcup_{f(x)\in D} V_x \quad \subset \quad D$$

so that D is a union of open subsets and hence is open in D.■

(ii) If $b \geq 1$ the inverse image of $(-\infty, b)$ is all of $\mathbb{R}$ and hence is open. If $1 > b \geq 0$ the inverse image of $(-\infty, b)$ is of $\mathbb{R} - [a, b]$ and hence is open. If $0 > b \geq 0$ the inverse image of $(-\infty, b)$ is of empty and hence is open. — These combine to show that the inverse of every $\mathcal{U}$-open subset is open.■

(iii) Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. If $f(t) < f(x) + \varepsilon$ for all $t \in \mathbb{R}$, then for every $\delta > 0$ we know that $|t - x| < \delta$ implies $f(t) < f(x) + \varepsilon$. If If $f(t_0) \geq f(x) + \varepsilon$ for some t_0, let b be the greatest lower bound of all t such that $f(t) \geq f(x) + \varepsilon$. Then the one-sided continuity condition implies that $f(b) \geq f(x) + \varepsilon$. Since f is increasing we must have $x < b$. Therefore, if $|t - x| < b - x$ then $f(t) < f(x) + \varepsilon$.■

(iv) Let $d \in \mathbb{R}$. Straightforward computation implies that $d > f(x) = ax + b$ if and only if

$$x \quad < \quad \frac{d - b}{a}$$

and hence inverse images of $\mathcal{U}$-open subsets are $\mathcal{U}$-open.■

(v) The inverse image of $(-\infty, 0)$ is $(0, +\infty)$, which is not $\mathcal{U}$-open, and therefore f is not continuous with respect to the $\mathcal{U}$ topology on $\mathbb{R}$ for both the domain and co-domain.■

2. (i) If $a < b$ then $(-\infty, a)$ is such a subset.■

(ii) Every nonempty subset has the form $\mathbb{R}$ or $(-\infty, c)$ for some c, and such sets clearly have the property described in this part of the exercise. — To see that $\mathbb{R} - \{x\}$ is not open, notice that $x + 1$ lies in the latter but x does not, so the condition in (i) is not satisfied and hence the set $\mathbb{R} - \{x\}$ cannot be $\mathcal{U}$-open.■

8. Continuity in topological spaces; bases

Exercises from Sutherland

`See the next two pages.`

Solutions to Chapter 8 exercises

8.1 (a) In this case the inverse image of any open set is itself, hence it is open, and f is continuous.

(b) Let the constant value be $y_0 \in Y$. Then $f^{-1}(U) = X$ if $y_0 \in U$ and $f^{-1}(U) = \emptyset$ if $y_0 \notin U$. In either case $f^{-1}(U)$ is open, and f is continuous.

(c) In this case $f^{-1}(U)$ is open in X no matter which set $U \subseteq Y$ is chosen, so f is continuous.

(d) In this case the only open sets in Y are $Y, \emptyset$. Now $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$. So f is continuous since both $\emptyset$ and X are open in X.

8.2 First suppose that $f : X \to Y$ is continuous, and let $x_0 \in X$. Let U be any open set in Y containing $f(x_0)$. Then $f^{-1}(U)$ is an open set in X containing x_0 and $f(f^{-1}(U)) \subseteq U$, so f is continuous at x_0. This holds for any $x_0 \in X$, so f is continuous at every point of X.

Now suppose that f is continuous at every point of X, and let $U \subseteq Y$ be open in Y. We want to show that $f^{-1}(U)$ is open in X. Let $x \in f^{-1}(U)$. Then $f(x) \in U$, and since f is continuous at x there exists an open set, call it V_x, open in X with $x \in V_x$ and $f(V_x) \subseteq U$, so $V_x \subseteq f^{-1}(U)$. Then

$$f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} V_x,$$

since each $x_0 \in f^{-1}(U)$ is in $V_{x_0} \subseteq \bigcup_{x \in f^{-1}(U)} V_x$, and conversely any point in $\bigcup_{x \in f^{-1}(U)} V_x$ is in V_{x_0} for some $x_0 \in f^{-1}(U)$, and $V_{x_0} \subseteq f^{-1}(U)$. Hence $\bigcup_{x \in f^{-1}(U)} V_x \subseteq f^{-1}(U)$. Now $f^{-1}(U)$ is a union of open sets, hence is open and f is continuous.

8.3 Suppose first that A is open in X. The only open sets in $\mathbb{S}$ are $\emptyset$, $\mathbb{S}$ and $\{1\}$. Now $\chi_A^{-1}(\emptyset) = \emptyset$, $\chi_A^{-1}(\mathbb{S}) = X$ and $\chi_A^{-1}(1) = A$, all of which are open in X so χ_A is continuous.

Conversely suppose that χ_A is continuous. Then A is open in X since $A = \chi_A^{-1}(1)$ and $\{1\}$ is open in $\mathbb{S}$.

8.4 Equivalence of topological spaces is reflexive, since for any space X the identity function of X is a homeomorphism. It is symmetric since if $f : X \to Y$ is a homeomorphism of topological spaces then so is $f^{-1} : Y \to X$. It is transitive since if $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms then so is $g \circ f : X \to Z$: for $g \circ f$ is bijective since f and g are, and continuous by Proposition 8.4; so is its inverse $f^{-1} \circ g^{-1}$ by definition and Proposition 8.4. Hence equivalence of topological spaces is an equivalence relation.

8.5 We need to show that any open set $U \subseteq \mathbb{R}$ is a union of finite open intervals. Now by definition of the usual topology, for any $x \in U$ there is some $\varepsilon_x > 0$ such that $(x - \varepsilon_x,\, x + \varepsilon_x) \subseteq U$. It is straightforward to check that

$$U = \bigcup_{x \in U} (x - \varepsilon_x,\, x + \varepsilon_x).$$

8.6 Suppose that the given condition holds. Then the inverse image under f of any finite open interval (a, b) is open, since $f^{-1}(a, b) = f^{-1}((-\infty, b) \cap (a, \infty)) = (f^{-1}(-\infty, b)) \cap (f^{-1}(a, \infty))$, which is the intersection of two sets open in X, and hence is open in X. Continuity of f now follows from Exercise 8.5 and Proposition 8.12.

8.7 We have to show that $\mathcal{B}$ is a basis, and that it is countable. We first show $\mathcal{B}$ is a basis. For this we need to show that any open subset U of $\mathbb{R}^2$ is the union of a subfamily of $\mathcal{B}$.

So let U be an open subset of $\mathbb{R}^2$ and let $(x, y) \in U$. It is enough to show that there is a set $B \in \mathcal{B}$ such that $(x, y) \in B \subseteq U$. First, there exists $\varepsilon > 0$ such that $B_{3\varepsilon}((x, y)) \subseteq U$. Now choose some rational number q such that $\varepsilon < q < 2\varepsilon$. Let q_1, q_2 be rational numbers with $|x - q_1| < \varepsilon/\sqrt{2}$ and $|y - q_2| < \varepsilon/\sqrt{2}$. Let us write d for the Euclidean distance in $\mathbb{R}^2$. Then

$$d((q_1, q_2), (x, y)) = \sqrt{(x - q_1)^2 + (y - q_2)^2} < \varepsilon.$$

Now $(x, y) \in B_\varepsilon((q_1, q_2)) \subseteq B_q((q_1, q_2)) \in \mathcal{B}$. Also, $B_q((q_1, q_2)) \subseteq B_{3\varepsilon}((x, y)) \subseteq U$: for if $(x', y') \in B_q(q_1, q_2)$ then

$$d((x', y'), (x, y)) \leqslant d((x', y'), (q_1, q_2)) + d((q_1, q_2), (x, y)) < q + \varepsilon < 3\varepsilon.$$

This shows that U is a union of sets in $\mathcal{B}$.

To show that $\mathcal{B}$ is countable, note that there is an injective function from $\mathcal{B}$ to $\mathbb{Q}^3$ defined by $B_q((q_1, q_2)) \mapsto (q, q_1, q_2)$. Now $\mathcal{B}$ is countable by standard facts about countable sets: $\mathbb{Q}$ is countable, a finite product of countable sets is countable, and any set from which there is an injective function to a countable set is countable.

1. Suppose that $f : X \to Y$ and $g : Y \to Z$ are {open/closed} mappings, and suppose that A is {open/closed} in X. Since f is a/an {open/closed} mapping, it follows that $f[A]$ is {open/closed} in Y, and since g is a/an {open/closed} mapping, it follows that $g \circ f[A] = g\big[f[A]\big]$ is {open/closed} in Z.■

2. The map j_X is continuous if and only if every $\mathcal{T}_2$-open set is also $\mathcal{T}_1$-open, which is true if and only if $\mathcal{T}_2$ is contained in $\mathcal{T}_1$. Similarly, the map j_X is open if and only if every $\mathcal{T}_1$-open set is also $\mathcal{T}_2$-open, which is true if and only if $\mathcal{T}_1$ is contained in $\mathcal{T}_2$.■

Note. If $\mathcal{F}_i$ is the family of closed sets associated to $\mathcal{T}_i$ we also have the following analog: The identity map j_X is continuous if and only if $\mathcal{F}_2$ is contained in $\mathcal{F}_1$, and j_X is closed if and only if $\mathcal{F}_1$ is contained in $\mathcal{F}_2$.■

3. To show that F is 1–1 onto we need to show that for each choice of u and v there is a unique solution to the system of equations

$$u \;=\; xe^y + y, \qquad v \;=\; xe^y - y \;.$$

Here is an elementary way of doing so. Subtacting the second equation from the first shows that $y = (u - v)/2$, and adding the two equations together yields

$$2\,x\,e^y \;=\; u + v \;.$$

Since we can solve uniquely for y, this equation shows that we can also solve uniquely for x, and these solutions are continuous functions of u and v. Therefore F is 1–1 onto, and we have derived an explicit description for the inverse mapping which shows that this inverse is continuous.■

4. Use the same approach as in the previous problem, finding unique solutions to the system of equations

$$(u, v, w) \;=\; \left(\frac{x}{2+y^2} + ye^z, \frac{x}{2+y^2} - ye^z, 2ye^z + z\right)$$

amounts to showing that for each choice of u, v and w there is a unique solution (x, y, z) for the displayed vector equation (which is a system of three scalar equations), and here is a summary of how this can be done: Subtracting the second equation from the first yields $2\,y\,e^z = u - v$, and by the third equation the left hand side is equal to $w - z$. Therefore we can solve for z uniquely in terms of u, v and w. If we substitute this result into $2\,y\,e^z = u - v$ we also get a unique solution for y in terms of u, v and w. Finally, if we add the original first and second equations we obtain $u + v = 2\,x/(2 + y^2)$. Since we already know that we can solve uniquely for y, this equation implies that we also get a unique solution for x terms of u, v and w. For each coordinate x, y, z the formulas for these coordinates in terms of u, v, w are continuous functions of the latter, and this implies that the inverse is continuous.■

5. (*i*) The function $f(x) = e^x + x$ is differentiable and it is strictly increasing because $f'(x) = e^x + 1$ is always positive. Since the limits of e^x and x as $x \to +\infty$ are equal to $+\infty$, the same is true for the limit of $f(x)$ as $x \to \infty$. Furthermore, since the limits of e^x and x as $x \to -\infty$ are equal to 0 and $-\infty$ respectively, the limit of $f(x)$ as $x \to -\infty$ is equal to $-\infty$, and therefore by the Intermediate Value Property the continuous mapping f is 1–1 onto from $\mathbb{R}$ to itself. If h is the inverse function to f, the standard rule for differentiation of inverses implies that if $y = f(x)$, then $h'(y) = 1/f'(x)$, so h is differentiable and hence continuous.■

(ii) Once again, the idea is to solve $(u, v) = (xe^y + y,\ xe^y - y)$ for x and y, and to show that the formulas for x and y are continuous functions of u and v. We have $u - v = 2y$ and hence $y = \frac{1}{2}(u - v)$. Furthermore, we also have $u + v = e^x + x$, so if h is the inverse function from (i), we have $x = h(u + v)$. As before, this shows that F is 1–1 onto and has a continuous inverse.■

Note. The inverse function to $f(x) = e^x + x$ cannot be expressed in terms of the standard functions from first year calculus. Further information on this inverse — and several closely related inverse functions — is summarized in the course directory file `lambert-fcn.pdf`; the latter also gives a reference for the key step in proving that there is no nice formula (in the sense of first year calculus) for the inverse function to f .

The preceding discussion reflects an unpleasant fact about inverse functions: *Even if a function with an inverse is defined by a nice formula, there is no guarantee that the inverse can be defined by a formula which is also reasonably nice.* — This is even true for polynomial functions. One example of this type is mentioned in Section II.3 of the course directory file `transcendentals.pdf`; however, the discsussion of the example involves material (Galois theory) which is not covered in this course or its prerequisites.