SOLUTIONS TO EXERCISES FOR

MATHEMATICS 145A — Part 4

Winter 2014

9. Some concepts in topological spaces

Exercises from Sutherland

See the next three pages.

Solutions to Chapter 9 exercises

9.1 The complement of any subset V of a discrete space X is open in X, so V is closed in X.

9.2 In order to be open in X, a subset either has to be empty or to have finite complement in X. So the subsets of X which are closed in X are X itself and any finite subset of X.

9.3 We may choose for example $U = (0,\, 1) \cup (2,\, 4)$, $V = (1,\, 3)$. Then

$$U \cap \overline{V} = (2,\, 3], \quad \overline{U} \cap V = [2,\, 3), \quad \overline{U} \cap \overline{V} = \{1\} \cup [2,\, 3], \quad \overline{U \cap V} = \emptyset.$$

9.4(a) Using Exercise 9.2, we see that any finite subset A of X is closed in X and hence is its own closure.

(b) Again using Exercise 9.2 we see that when A is infinite, the smallest closed set containing A is X. Hence $\overline{A} = X$ (by Proposition 9.10 (f)).

9.5 (a) This is false in general. For a counterexample, let X be the space $\{0,\, 1\}$ with the discrete topology, let Y be the space $\{0,\, 1\}$ with the indiscrete topology, and let f be the identity function. Then f is continuous for example by Exercise 8.1, (c) or (d). Also, $A = \{0\}$ is closed in X but $f(A) = A$ is not closed in Y. (The same counterexample would work for any set with at least two points in it.)

(b) Again this is false: we have seen a counterexample in Exercise 9.3 - take $A = (0,\, 1) \cup (2,\, 4)$ and $B = (1,\, 3)$ in $\mathbb{R}$, and we have $A \cap \overline{B} = (2,\, 3]$ while $\overline{A \cap B} = \emptyset$.

(c) This is false too. Let f be as in the counterexample for (a) and let $B = \{0\}$. Then $\overline{B} = \{0,\, 1\}$ so $f^{-1}(\overline{B}) = \{0,\, 1\}$ but $f^{-1}(B) = \{0\}$ so $\overline{f^{-1}(B)} = \{0\}$.

9.6 (a) For each $j = 1,\, 2,\, \ldots,\, m$ we have $A_j \subseteq \bigcup_{i=1}^{m} A_i$ so $\overline{A_j} \subseteq \overline{\bigcup_{i=1}^{m} A_i}$. Hence $\bigcup_{i=1}^{m} \overline{A_i} \subseteq \overline{\bigcup_{i=1}^{m} A_i}$.

Conversely, since each A_i is closed in X and a finite union of closed sets is closed, $\bigcup_{i=1}^{m} \overline{A_i}$ is closed in X. Also, since $A_i \subseteq \overline{A_i}$, we have $\bigcup_{i=1}^{m} A_i \subseteq \bigcup_{i=1}^{m} \overline{A_i}$. So the latter is a closed subset of X containing $\bigcup_{i=1}^{m} A_i$ and by Proposition 9.10 (f) it contains $\overline{\bigcup_{i=1}^{m} A_i}$. Hence $\bigcup_{i=1}^{m} \overline{A_i} = \overline{\bigcup_{i=1}^{m} A_i}$.

(b) The right-hand side is an intersection of sets closed in X, hence is closed in X. It also contains $\bigcap_{i \in I} A_i$ since $A_i \subseteq \overline{A_i}$ for each $i \in I$. The required result follows by Proposition 9.10 (f).

9.7 Suppose first that $f : X \to Y$ is continuous and that $A \subseteq X$. Let $y \in f(\overline{A})$, say $y = f(x)$ where $x \in \overline{A}$. Let U be any open subset of Y containing y. Then $f^{-1}(U)$ is open in X and $x \in f^{-1}(U)$. Hence there exists $a \in A \cap f^{-1}(U)$, and $f(a) \in U$. Hence $y \in \overline{f(A)}$. This shows that $f(\overline{A}) \subseteq \overline{f(A)}$.

Conversely suppose that $f(\overline{A}) \subseteq \overline{f(A)}$ for any subset $A \subseteq X$. In particular we apply this with $A = f^{-1}(V)$ where V is closed in Y. Then $f(\overline{f^{-1}(V)} \subseteq \overline{f(f^{-1}(V)} \subseteq \overline{V} = V$. Hence $\overline{f^{-1}(V)} \subseteq f^{-1}(V)$. Since always $f^{-1}(V) \subseteq \overline{f^{-1}(V)}$, we have $\overline{f^{-1}(V)} = f^{-1}(V)$ hence by Proposition 9.10 (c) $f^{-1}(V)$ is closed in X, showing that f is continuous.

9.8 (a) When A is finite, the only open set contained in A is $\emptyset$, so $\mathring{A} = \emptyset$. In this case, as we saw in Exercise 9.4, $\overline{A} = A$. Hence $\partial A = A$.

(b) Suppose that A is infinite. We distinguish two cases.

Case (1) If $X \setminus A$ is finite then A is open in X so $\mathring{A} = A$.

Case (2) If $X \setminus A$ is infinite, then $X \setminus B$ is infinite for any subset B of A, so $\emptyset$ is the only subset of A which is open in X. Hence in this case $\mathring{A} = \emptyset$.

Since (see Exercise 9.4) $\overline{A} = X$ when A is infinite, in Case (1) $\partial A = X \setminus A$ while in Case (2) $\partial A = X$.

9.9 (a) If $a \in \mathring{A}$ then by definition there is some open set U of X such that $a \in U \subseteq A$. In particular then $a \in A$. So $\mathring{A} \subseteq A$.

(b) if $A \subseteq B$ and $x \in \mathring{A}$ then by definition there is some open subset U of X such that $a \in U \subseteq A$. Since $A \subseteq B$ then also $U \subseteq B$, so $a \in \mathring{B}$. This proves that $\mathring{A} \subseteq \mathring{B}$.

(c) If A is open in X then for every $a \in A$ there is an open set U (namely $U = A$) such that $a \in U \subseteq A$, so $a \in \mathring{A}$. This shows $A \subseteq \mathring{A}$, and together with (a) we get $\mathring{A} = A$.

Conversely if $\mathring{A} = A$ then for every $a \in A$ there exists a set open in X, call it U_a, such that $a \in U_a \subseteq A$. It is straightforward to check that $A = \bigcup_{a \in A} U_a$ which is a union of sets open in X, so is open in X.

(d) by (a), the interior of $\mathring{A}$ is contained in $\mathring{A}$. Conversely suppose that $a \in \mathring{A}$. Then there exists a subset U open in X such that $a \in U \subseteq A$. Now for any point $x \in U$ we have $x \in U \subseteq A$, so also $x \in \mathring{A}$. This shows that $a \in U \subseteq \mathring{A}$, so a is in the interior of $\mathring{A}$. These together show that the interior of $\mathring{A}$ is $\mathring{A}$.

(e) This follows from (c) and (d).

(f) We know that $\mathring{A}$ is open in X from (e). Suppose that B is open in X and that $B \subseteq A$. By (b) then $\mathring{B} \subseteq \mathring{A}$. Since B is open we have $\mathring{B} = B$ by (c). So $B \subseteq \mathring{A}$, which says that $\mathring{A}$ is the largest open subset of X contained in A.

9.10 Suppose first that $f : X \to Y$ is continuous, and let $B \subseteq Y$. Since $\mathring{B}$ is open in Y, by continuity $f^{-1}(\mathring{B})$ is open in X, so since also $f^{-1}(\mathring{B}) \subseteq f^{-1}(B)$ we have that $f^{-1}(\mathring{B})$ is contained in the interior of $f^{-1}(B)$.

Conversely suppose that for every subset $B \subseteq Y$ we have $f^{-1}(\mathring{B})$ is contained in the interior of $f^{-1}(B)$. We apply this with B open in Y, when $\mathring{B} = B$ so we get that $f^{-1}(B)$ is contained in the interior of $f^{-1}(B)$, which says that $f^{-1}(B)$ is open in X. Hence f is continuous.

9.11 Since $\mathring{A}_i \subseteq A_i$ we get $\bigcap_{i=1}^{m} \mathring{A}_i \subseteq \bigcap_{i=1}^{m} A_i$. Also, $\bigcap_{i=1}^{m} \mathring{A}_i$ is the intersection of a finite family of open sets, so is open in X, hence it is contained in the interior of $\bigcap_{i=1}^{m} A_i$. Conversely, $\bigcap_{i=1}^{m} A_i \subseteq A_j$ for each $j \in \{1, 2, \ldots, m\}$; it follows that the interior of $\bigcap_{i=1}^{m} A_i$ is contained in $\mathring{A}_j$ for each $j \in \{1, 2, \ldots, m\}$ so the interior of $\bigcap_{i=1}^{m} A_i$ is contained in $\bigcap_{i=1}^{m} \mathring{A}_i$. This proves the result.

9.12 This follows since $\bigcup_{i \in I} \mathring{A}_i$ is open and contained in $\bigcup_{i=1}^{m} A_i$.

An example: take $X = \mathbb{R}$, $A_1 = (0, 1)$, $A_2 = [1, 2)$. Then $\mathring{A}_1 \cup \mathring{A}_2 = (0, 1) \cup (1, 2)$ while the interior of $A_1 \cup A_2$ is $(0, 2)$.

9.13 This follows from the fact that $\partial A = \overline{A} \cap \overline{X \setminus A}$ (Proposition 9.20) since each of $\overline{A}$, $\overline{X \setminus A}$ is closed in X hence so is their intersection.

9.14 (a) If A is closed in X then $\overline{A} = A$, so $\partial A = \overline{A} \setminus \mathring{A} \subseteq A$.
By definition $\partial A = \overline{A} \setminus \mathring{A}$ so in general $\overline{A} = \mathring{A} \cup \partial A$. So if $\partial A \subseteq A$ then both ∂A and $\mathring{A}$ are subsets of A so $\overline{A} \subseteq A$ and A is closed in X.

9.14 (b) Suppose that $\partial A = \emptyset$. This says that $\mathring{A} = \overline{A}$, and since always $\mathring{A} \subseteq A \subseteq \overline{A}$ we get $A = \mathring{A}$ and $A = \overline{A}$. From the first of these A is open and from the second A is closed in X.
Conversely if A is both open and closed in X then $A = \mathring{A}$ and $A = \overline{A}$. Hence $\partial A = \overline{A} \setminus \mathring{A} = \emptyset$.

9.15 Since $\partial A = \overline{A} \setminus \mathring{A}$, certainly $\partial A \cap \mathring{A} = \emptyset$. By the definition $\partial A = \overline{A} \setminus \mathring{A}$ we know that $\partial A \subseteq \overline{A}$, and $\mathring{A} \subseteq A \subseteq \overline{A}$. So the disjoint union $\partial A \sqcup \mathring{A} \subseteq \overline{A}$. Conversely since $\partial A = \overline{A} \setminus \mathring{A}$, we have $\overline{A} \subseteq \mathring{A} \sqcup \partial A$. These two together show that $\overline{A} = \mathring{A} \sqcup \partial A$.
Now if $B \subset X$ and $B \cap A \neq \emptyset$ then $B \cap \overline{A} \neq \emptyset$ so either $B \cap \mathring{A} \neq \emptyset$ or $B \cap \partial A \neq \emptyset$.

9.16 First $\mathring{A} \cap (X \mathring{\setminus} A) = \emptyset$ since $\mathring{A} \subseteq A$ and $(X \mathring{\setminus} A) \subseteq X \setminus A$. Exercise 9.15 shows that $\mathring{A} \cap \partial A = \emptyset$. Since $\partial A = \partial(X \setminus A)$ by Corollary 9.21, $\partial A \cap (X \mathring{\setminus} A) = \partial(X \setminus A) \cap (X \mathring{\setminus} A) = \emptyset$. Thus the three sets are pairwise disjoint. To see that their union is X, we use Exercise 9.15 and the fact that $\partial A = \partial(X \setminus A)$ (Corollary 9.21):
$\mathring{A} \cup \partial A \cup (X \mathring{\setminus} A) = \mathring{A} \cup \partial A \cup \partial A \cup (X \mathring{\setminus} A) = \mathring{A} \cup \partial A \cup \partial(X \setminus A) \cup (X \mathring{\setminus} A) = \overline{A} \cup \overline{X \setminus A} = X$.

Additional exercise(s)

1. If f is continuous, then for each open set in the family $\mathcal{V}$ the inverse image $f^{-1}[V]$ is automatically open.

Conversely, since $\mathcal{V}$ generates the topology on Y, every open subset W in Y is a union of finite intersections as below, where each $V_{\alpha,j} \in \mathcal{V}$:

$$W \quad = \quad \bigcup_{\alpha} \left(V_{\alpha,1} \cap \ \cdots \ V_{\alpha,k(\alpha)} \right)$$

Since the inverse image construction sends unions to unions and intersections to intersections, we have

$$f^{-1}[W] \quad = \quad \bigcup_{\alpha} \left(f^{-1}[V_{\alpha,1}] \cap \ \cdots \ f^{-1}[V_{\alpha,k(\alpha)}] \right)$$

and this is open because we assumed each set $f^{-1}[V_{\alpha,j}]$ is open.∎

2. No points with $|v| < 1$ can lie in the boundary of either set because such points are in the interiors of both D^2 and $N_1(0)$, and not points with $|v| > 1$ can lie in the boundary of either set because such points are in the interiors of both $\mathbb{R}^2 - D^2$ and $\mathbb{R}^2 - N_1(0)$, Thus a boundary point for either set must lie on the unit circle S^1.

By the preceding observations, it will suffice to show that if $v \in S^1$ then there are sequences $\{a_n\}$ in $\mathbb{R}^2 - D^2$ and $\{b_n\}$ in $N_1(0)$ whose limits are equal to v. We can do this very easily by taking $a_n = \left(1 + \frac{1}{n}\right) \cdot v$ and $b_n = \left(1 - \frac{1}{n}\right) \cdot v$.∎

3. The two displayed sets in the exercise are open, for if $H(x, y) = y - f(x)$, then H is continuous and the sets in question are the inverse images of $(-\infty, 0)$ and $(0, \infty)$ respectively. If we denote these inverse images by W_- and W_+ respectively, then $\mathbb{R}^2$ is the union of the pairwise disjoint subsets W_-, Γ_f and W_+.

Clearly the closures of the open sets W_- and W_+ are contained in $W_- \cup \Gamma_f$ and $W_+ \cup \Gamma_f$ respectively, so it follows that the boundaries of the open sets W_- and W_+ must be contained in Γ_f. To conclude the proof, we need to show that very point of the graph is a limit point of each open subset.

We can do this by the same sort of argument which was used in the preceding exercise. Specifically, the limits of the sequences in W_- and W_+ given by $a_n = \left(x, f(x) - \frac{1}{n}\right)$ and $b_n = \left(x, f(x) + \frac{1}{n}\right)$ are both equal to $(x, f(x))$.∎

10. Subspaces and product spaces

Exercises from Sutherland

`See the next six pages.`

Solutions to Chapter 10 exercises

10.1 The subspace topology $\mathcal{T}_A$ consists of all sets $U \cap A$ where $U \in \mathcal{T}$. Hence $\mathcal{T}_A = \{\emptyset, \{a\}, A\}$.

10.2 (T1) Since X, $\emptyset \in \mathcal{T}$, the family $\mathcal{T}_A$ contains $\emptyset \cap A = \emptyset$ and $X \cap A = A$.

(T2) Suppose that $V_1, V_2 \in \mathcal{T}_A$. Then $V_1 = A \cap U_1$ and $V_2 = A \cap U_2$ for some $U_1, U_2 \in \mathcal{T}$. Hence $V_1 \cap V_2 = (A \cap U_1) \cap (A \cap U_2) = A \cap (U_1 \cap U_2)$. But $U_1 \cap U_2 \in \mathcal{T}$ since $\mathcal{T}$ is a topology, so $V_1 \cap V_2 \in \mathcal{T}_A$.

(T3) Suppose that $V_i \in \mathcal{T}_A$ for all i in some indexing set I. Then for each $i \in I$ there exists some $U_i \in \mathcal{T}$ such that $V_i = A \cap U_i$. Then

$$\bigcup_{i \in I} V_i = \bigcup_{i \in I} A \cap U_i = A \cap \bigcup_{i \in I} U_i$$

and since $\mathcal{T}$ is a topology, $\bigcup_{i \in I} U_I$ is in $\mathcal{T}$, so $\bigcup_{i \in I} V_i$ is in $\mathcal{T}_A$.

10.3 We have to show that the subspace topology $\mathcal{T}_A$ on A is the same as the co-finite topology on A. First suppose $V \subseteq A$ is in the cofinite topology for A. Either $V = \emptyset$, and then $V = A \cap \emptyset \in \mathcal{T}_A$, or $A \setminus V$ is finite. In this latter case, let $U = (X \setminus A) \cup V$ Then $A \cap U = V$, and U is in the co-finite topology for X, since $X \setminus U = A \setminus V$ which is finite.

Conversely suppose that $V = A \cap U$ where U is in the co-finite topology $\mathcal{T}$ for X. Then either $U = \emptyset$, so $A \cap U = \emptyset$, and V is in the co-finite topology for A, or $X \setminus U$ is finite, in which case $A \setminus V \subseteq X \setminus U$ is finite and again V is in the co-finite topology for A.

10.4 First we show that any subset $V \in \mathcal{T}_A$ is d_A-open. Suppose that $a \in V$. We want to show that there exists $\varepsilon > 0$ with $B_\varepsilon^{d_A}(a) \subseteq V$. Now $V \in \mathcal{T}_A$ so $V = A \cap U$ for some U open in X. Then $a \in U$, and U is open in X (i.e. $U \in \mathcal{T} = \mathcal{T}_d$) so there exists $\varepsilon > 0$ such that $B_\varepsilon^d(a) \subseteq U$. But $B_\varepsilon^{d_A}(a) = A \cap B_\varepsilon^d(a)^\dagger$, so $B_\varepsilon^{d_A}(a) \subseteq A \cap U = V$ as required.

Proof of †. If $x \in B_\varepsilon^{d_A}(a)$ then $x \in A$ and $d(x, a) = d_A(x,\ a) < \varepsilon$ so $x \in A \cap B_\varepsilon(A)$. If $x \in A \cap B_\varepsilon(a)$ then $x \in A$ and $d_A(x, a) = d(x, a) < \varepsilon$, so $x \in B_\varepsilon^{d_A}(a)$. Together these show that $B_\varepsilon^{d_A}(a) = A \cap B_\varepsilon^d(a)$.

Conversely we wish to show that any d_A-open subset V of A is in $\mathcal{T}_A$. For each $a \in V$ there exists $\varepsilon_a > 0$ with $B_{\varepsilon_a}^{d_A}(a) \subseteq V$. Let $U = \bigcup_{a \in A} B_{\varepsilon_a}^d(a)$. Then U is open in X (as a union of open balls) and I claim that $V = A \cap U$. For

$$A \cap U = A \cap \bigcup_{a \in V} B_{\varepsilon_a}^d(a)(a) = \bigcup_{a \in V} A \cap B_{\varepsilon_a}^d(a) = \bigcup_{a \in V} B_{\varepsilon_a}^{d_A}(a) = V,$$

where the last equality is straightforward to check. This shows that $V \in \mathcal{T}_A$ as required.

10.5 Since V is closed in X its complement $X \setminus V$ is open in X. Now $A \setminus (V \cap A) = A \cap (X \setminus V)$ by Exercise 2.2. So $A \setminus (V \cap A) \in \mathcal{T}_A$. This shows that $V \cap A$ is closed in $(A, \mathcal{T}_A)$.

10.6 (a) Suppose that $W \in \mathcal{T}_A$ and that $A \in \mathcal{T}$. Now $W = A \cap U$ for some $U \in \mathcal{T}$ since $W \in \mathcal{T}_A$. Then since also $A \in \mathcal{T}$ we have $W = A \cap U \in \mathcal{T}$ as required.

(b) We have $X \setminus A \in \mathcal{T}$ and $A \setminus W \in \mathcal{T}_A$ so $A \setminus W = U \cap A$ for some $U \in \mathcal{T}$. So

$$X \setminus W = (X \setminus A) \cup (A \setminus W) = (X \setminus A) \cup (U \cap A) = (X \setminus A) \cup U,$$

where the last equality follows since $U = (U \cap A) \cup (U \cap (X \setminus A))$ and $U \cap (X \setminus A) \subseteq X \setminus A$. So $X \setminus W$ is open in X, as the union of open sets, so W is closed in X.

10 .7(a) We use Proposition 3.13: for any subset $B \subseteq Y$ we have

$$f^{-1}(B) = \bigcup_{i \in I} (f|U_i)^{-1}(B).$$

Now let B be open in Y. For each $i \in I$, continuity of $f|U_i$ implies that $(f|U_i)^{-1}(B)$ is open in U_i and hence, by Exercise 10.6 (a), it is open in X. Hence $f^{-1}(B)$ is a union of sets open in X, so it is open in X and f is continuous as required.

(b) We again use Proposition 3.13: for any subset $B \subseteq Y$ we have

$$f^{-1}(B) = \bigcup_{i=1}^{n} (f|V_i)^{-1}(B).$$

Now suppose B is closed in Y. Then continuity of $f|V_i$ implies that $(f|V_i)^{-1}(B)$ is closed in V_i and hence, by Exercise 10.6 (b), it is closed in X. Hence $f^{-1}(B)$ is the union of a finite number of sets closed in X, so it is closed in X, and f is continuous as required.

10.8 First let V be any subset of A which is in $\mathcal{T}_A$. Then $V = A \cap U$ for some U open in X. Let $W = B \cap U$. Then W is in $\mathcal{T}_B$, and $V = A \cap U = B \cap (A \cap U) = A \cap (B \cap U) = A \cap W$. So V is in the topology on A induced by $\mathcal{T}_B$.

Conversely suppose that $V \subseteq A$ is in the topology on A induced by $\mathcal{T}_B$. Then $V = A \cap W$ for some $W \in \mathcal{T}_B$, and by definition of $\mathcal{T}_B$ we know that $W = B \cap U$ for some $U \in \mathcal{T}$. Since $V = A \cap W = A \cap (B \cap U) = (A \cap B) \cap U = A \cap U$, it follows that $V \in \mathcal{T}_A$ as required.

10.9 (a) First suppose $x \in B_1$. Then $x \in X_1$, and also for any set W open in X_1 with $x \in W$ we have $W \cap A \neq \emptyset$. Now let U be any set open in X_2 with $x \in U$. Then $W = U \cap X_1$ is open in X_1 and contains x, so $W \cap A \neq \emptyset$. Then $U \cap A = U \cap (A \cap X_1) = (U \cap X_1) \cap A = W \cap A \neq \emptyset$, so $x \in B_2$. Since also $x \in X_1$ this shows that $B_1 \subseteq B_2 \cap X_1$.

Conversely suppose that $x \in B_2 \cap X_1$. Then $x \in X_1$ and for any subset U open in X_2 with $x \in U$ we know $U \cap A \neq \emptyset$. Now let W be an open subset of X_1 with $x \in W$. Then $W = X_1 \cap U$ for some U open in X_2 with $x \in U$. Hence $U \cap A \neq \emptyset$, so since $A \subseteq X_1$ we have $U \cap A = U \cap X_1 \cap A = W \cap A$, so $W \cap A \neq \emptyset$, showing that $x \in B_1$.

Taking these two together we have $B_1 = B_2 \cap X_1$.

(b) If X_1 is closed in X_2 then B_1, being closed in X_1, is closed in X_2 by Exercise 10.6 (b). Now B_1 is a closed subset of X_2 containing A and hence containing B_2. Since we already know from (a) above that $B_1 \subseteq B_2$ we get $B_1 = B_2$.

10.10 Let $f^{-1} : Y \to X$ be the (continuous) inverse function of f. Then $f^{-1}|B : B \to X$ is continuous, by Corollary 10.5. Since $f^{-1}|B$ maps B onto A, it defines a continuous function from B to A (by Proposition 10.6). In fact this function is inverse to the map $g : A \to B$ defined by f, and shows that g is a homeomorphism from A to B.

Since f is one-one onto Y, and $f(A) = B$, we have also that $f(X \setminus A) = Y \setminus B$, so f defines a one-one onto map $h : X \setminus A \to Y \setminus B$ which is a homeomorphism just as g is.

10.11 Any singleton set $\{(x,\, y)\}$ in $X \times Y$ is the product $\{x\} \times \{y\}$ of sets which are open in X, Y since they have the discrete topology, so $\{(x,\, y)\}$ is open in the product topology. Hence any subset of $X \times Y$ is open in the product topology, which is therefore discrete.

10.12 The open sets in $\mathcal{S}$ are $\emptyset$, $\mathcal{S}$, $\{1\}$, so a basis for the open sets in the product topology on $\mathcal{S} \times \mathcal{S}$ is $\{\{1\} \times \{1\},\, \{1\} \times \mathcal{S},\, \mathcal{S} \times \{1\},\, \mathcal{S} \times \mathcal{S}\}$. Thus the open sets in the product topology are:

$$\emptyset,\ \{(1,\,1)\}, \{(1,\,0),\,(1,\,1)\},\ \{(0,\,1),\,(1,\,1)\},\ \{(0,\,1),\,(1,\,0),(1,\,1)\},\ \mathcal{S} \times \mathcal{S}.$$

10.13 Consider the case when Y is infinite, and X contains at least two points. Then we may let U be a non-empty open subset of X with $U \neq X$. Then the complement of $U \times Y$ is $(X \setminus U) \times Y$, which is infinite. But $U \times Y \neq X \times Y$. Hence $U \times Y$ is not open in the co-finite topology on $X \times Y$ although it is open in the product of the co-finite topologies on X and Y.

10.14 Since the product metrics on $X \times Y$ in Example 5.10 are all topologically equivalent, it is enough to prove this with $d = d_\infty$. To prove that $\mathcal{T}_d$ coincides with the product topology of $(X,\, \mathcal{T}_X)$ and $(Y,\, \mathcal{T}_Y)$ it is enough to show that any set in a basis for one of these topologies is open in the other topology.

We use the basis for $\mathcal{T}_X$ consisting of all open balls $B_r^{d_X}(x)$ and similarly for $\mathcal{T}_Y$, and we use the basis for $\mathcal{T}_d$ the set of all open balls $B_s^d((x,\, y))$. We show first that any open ball in this basis for $\mathcal{T}_d$ is open in the product topology of $\mathcal{T}_X$ and $\mathcal{T}_Y$. This follows since $B_s^d((x,\, y)) = B_s^{d_X}(x) \times B_s^{d_Y}(y)$: for $d((x',\, y'),\, (x,\, y) < s$ iff both $d(x',\, x) < s$ and $d(y',\, y) < s$. But $B_s^{d_X}(x) \times B_s^{d_Y}(y)$ is a (basis) open set for the product of the topologies $\mathcal{T}_X$, $\mathcal{T}_Y$. So each $B_s^d((x\,,y))$ is open in the product topology of $(X,\, \mathcal{T}_X)$, $(Y,\, \mathcal{T}_Y)$.

Conversely suppose $U \times V$ is any basis set in the product topology, and let $(x,\, y) \in U \times V$. Then U is open in X, so there exists $r > 0$ such that $B_r^{d_X}(x) \subseteq U$. Similarly there exists $s > 0$ such that $B_s^{d_Y}(y) \subseteq V$. Let $t = \min\{r,\, s\}$. Then $B_t^d((x,\, y)) \subseteq U \times V$, since if $d((x',\, y'),\, (x,\ y) < t$ then both $d_X(x',\ x) < t \leqslant r$ and $d_Y(y',\, y) < t \leqslant s$, so $x' \in U$ and $y' \in V$.

10.15 (a) Any open subset W of $X \times Y$ is a union $\bigcup_{i \in I} U_i \times V_i$ for some indexing set I, where each U_i is open in X and each V_i is open in Y. We may as well assume that no V_i (and no U_i) is empty, since if it were then $U_i \times V_i$ would be empty, and hence does not contribute to the union. The point of this is that $p_X(U_i \times V_i) = U_i$ for all $i \in I$. Now

$$p_X(W) = p_X\left(\bigcup_{i \in I} U_i \times V_i\right) = \bigcup_{i \in I} p_X(U_i \times V_i) = \bigcup_{i \in I} U_i,$$

which is open in X as a union of open sets. Similarly $p_Y(W)$ is open in Y.

(b) Consider the set $W = \{(x, y) \in \mathbb{R} \times \mathbb{R} : xy = 1\}$. This is closed in $\mathbb{R} \times \mathbb{R}$: a painless way to see this is to consider the function $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $m(x, y) = xy$. Then m is continuous (by Propositions 8.3 and 5.17) and $\{1\}$ is closed in $\mathbb{R}$, so $W = m^{-1}(1)$ is closed in $\mathbb{R} \times \mathbb{R}$ by Proposition 9.5. But $p_1(W) = \mathbb{R} \setminus \{0\}$ is not closed in $\mathbb{R}$.

10.16 For use in (ii) and (iii) we check that for any subsets V, W of sets X, Y we have

$$(X \times Y) \setminus (V \times W) = \{X \times (Y \setminus W)\} \cup \{(X \setminus V) \times Y\}. \qquad (*)$$

For (x, y) is in the left-hand side iff either $y \notin W$ or $x \notin V$, and the same is true for the right-hand side.

(i) First suppose that (x, y) is in the interior of $A \times B$. Then there is some set W open in $X \times Y$ such that $(x, y) \in W \subseteq A \times B$. By definition of the product topology, there exist open subsets U of X and V of Y such that $(x, y) \in U \times V \subseteq W$. This shows that $x \in U \subseteq A$ and $y \in V \subseteq B$, so $x \in \mathring{A}$ and $y \in \mathring{B}$, hence $(x, y) \in \mathring{A} \times \mathring{B}$. This shows that the interior of $A \times B$ is contained in $\mathring{A} \times \mathring{B}$.

Conversely suppose that $(x, y) \in \mathring{A} \times \mathring{B}$. Then there exist sets U, V open in X, Y respectively such that $x \in U \subseteq A$ and $y \in V \subseteq B$. Then $U \times V$ is open in $X \times Y$ and $(x, y) \in U \times V \subseteq A \times B$ so (x, y) is in the interior of $A \times B$. Hence $\mathring{A} \times \mathring{B}$ is contained in the interior of $A \times B$.

Together these show that the interior of $A \times B$ is $\mathring{A} \times \mathring{B}$.

(ii) By (*), $X \times Y \setminus (\overline{A} \times \overline{B}) = \{(X \setminus \overline{A}) \times Y\} \cup \{X \times (Y \setminus \overline{B})\}$, the union of two open sets which is open in $X \times Y$ so $\overline{A} \times \overline{B}$ is closed in $X \times Y$. Since also $A \times B \subset \overline{A} \times \overline{B}$ it follows from Proposition 9.10 (f) that $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$.

Conversely suppose that $x \in \overline{A}$ and that $y \in \overline{B}$. Let W be any open subset of $X \times Y$ containing (x, y). Let U, V be open subsets of X, Y such that $(x, y) \in U \times V \subseteq W$. Since U contains a point $a \in A$ and V contains a point $b \in B$, it follows that W contains the point (a, b) of $A \times B$. Hence $(x, y) \in \overline{A \times B}$. This shows that $\overline{A} \times \overline{B} \subseteq \overline{A \times B}$.

Together these prove that $\overline{A \times B} = \overline{A} \times \overline{B}$.

(iii) This may be deduced from (i) and (ii). For using (*),

$$\partial(A \times B) = \overline{A \times B} \setminus (A \mathring{\times} B) = \overline{A} \times \overline{B} \setminus (\mathring{A} \times \mathring{B}) = ((\overline{A} \setminus \mathring{A}) \times \overline{B}) \cup (\overline{A} \times (\overline{B} \setminus \mathring{B})) = (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B).$$

10.17 First, t is continuous by Proposition 10.11, since if p_1, p_2 are the projections of $X \times X$ on the first, second factors, then $p_1 \circ t = p_2$ and $p_2 \circ t = p_1$, and p_2, p_1 are both continuous. Now we observe that t is self-inverse, so it is a homeomorphism.

10.18 From Proposition 10.12, $f \times g$ is continuous. Since both f and g are 1-1 onto it is easy to see that $f \times g$ is 1-1 onto. The inverse of $f \times g$ is $f^{-1} \times g^{-1}$. Now f^{-1}, g^{-1} are both continuous since f, g are homeomorphisms, so $f^{-1} \times g^{-1}$ is continuous, again by Proposition 10.12. Hence $f \times g$ is a homeomorphism.

10.19 (a) The graph of f is a curve through $(0, 1)$ which has the lines $x = -1$, $x = 1$ as vertical asymptotes. We argue as in Proposition 10.18: let $\theta : X \to G_f$ be defined by $\theta(x) = (x, f(x)$ and let $\phi : G_f \to X$ be defined by $\phi(x, f(x)) = x$. Then θ and ϕ are easily seen to be mutually inverse. Continuity of θ follows from Proposition 10.11 since $p_1 \circ \theta$ is the identity map of X and $p_2 \circ \theta$ is the continuous function f. Continuity of ϕ follows since ϕ is the restriction to G_f of the continuous projection $p_1 : X \times \mathbb{R} \to X$. Hence θ is a homeomorphism (with inverse ϕ).

(b) The graph of f is not easy to draw, but it oscillates up and down with decreasing amplitude as x approaches 0 from the right. Continuity of $f : [0, \infty) \to \mathbb{R}$ on $(0, \infty)$ follows by continutiy of the sine function together with Propositions 8.3 and 5.17. Continuity (from the right) at 0 follows from Exercise 4.14. Now again arguing as in Proposition 10.18 we see that $x \mapsto (x, f(x))$ defines a homeomorphism from $[0, \infty)$ to G_f.

10.20 Suppose first that the topology on X is discrete. Then as we saw in Exercise 10.11 the topology on $X \times X$ is also discrete, so any subset, in particular Δ, is open in $X \times X$.

Conversely suppose that Δ is open in the topological product $X \times X$. Then for any $x \in X$, $(x, x) \in \Delta$ and Δ is open, so there exist open subsets U, V of X such that $(x, x) \in U \times V \subseteq \Delta$. Then $x \in U$, and $x \in V$. Moreover, if any other point $y \in U$ we would have $(y, x) \in U \times V \subseteq \Delta$. But $(y, x) \notin \Delta$ since $y \neq x$. So $U = \{x\}$, and this says $\{x\}$ is open in X. Hence X has the discrete topology.

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1. Let $\overline{B}$ denote the closure of B in X. Since B is dense in A, the closure of B in A, which is $\overline{B} \cap A$ is equal to A, which means that $\overline{B} \supset A$. Since $\overline{B}$ is a closed subset containing A, we then have $\overline{B} \supset \overline{B} = X$, and hence B is dense in X.■

2. Follow the hint. The set of points in $\mathbb{R}^2$ such that $xy = 1$ is the zero set of the continuous real valued function $f(x,y) = xy-1$ and hence is closed in $\mathbb{R}^2$, but its image under either coordinate projection $\mathbb{R}^2 \to \mathbb{R}$ is $\mathbb{R} - \{(0,0)\}$, which is not closed in $\mathbb{R}$.■

3. The topologies $\mathcal{T}_X|A$ and $\mathcal{T}_Y|B$ consist (respectively) of all subsets of the form $U \cap A$ and $V \cap B$ where U is open in X and V is open in Y, so the product topology $(\mathcal{T}_X \mid A) \prod (\mathcal{T}_Y) \mid B$ is generated by all sets of the form $(U \cap A) \times (V \cap B)$ for such U and V.

Similarly, the subspace topology $(\mathcal{T}_X \prod \mathcal{T}_Y) \mid A \times B$ is generated by all sets of the form $(U \times V) \cap (A \times B)$, where U is open in X and V is open in Y.

Since $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$ by Exercise 2.5 in Sutherland, we see that both topologies are generated by the same family of subsets, and therefore the two topologies must coincide.■

4. (*i*) Suppose that $\mathcal{U}$ is a topology on Y such that $f : (X, \mathcal{T}_X) \to (Y, \mathcal{U})$ is continuous. Then $V \in \mathcal{U}$ implies that $f^{-1}[V] \in \mathcal{T}_X$, and therefore $\mathcal{U}$ is contained in $f_*\mathcal{T}_X$. To complete the proof, it will suffice to show that the latter defines a topology on Y. Clearly $\emptyset$ and Y belong to $f_*\mathcal{T}_X$ because their inverse images are the open sets $\emptyset$ and X respectively. Suppose now that $V_\alpha \in f_*\mathcal{T}_X$ for all $\alpha \in A$. Then for each α we have $f^{-1}[V_\alpha] \in \mathcal{T}_X$, and since $\mathcal{T}_X$ is a topology for X we know that

$$f^{-1}\left[\bigcup_{\alpha \in A} V_\alpha\right] \quad = \quad \bigcup_{\alpha \in A} f^{-1}[V_\alpha]$$

also belongs to $\mathcal{T}_X$, so that the union of the sets V_α belongs to $f_*\mathcal{T}_X$. Similarly, if V_1 and V_2 belong to $f_*\mathcal{T}_X$ we have $f^{-1}[V_i] \in \mathcal{T}_X$ for $i = 1, 2$, so that

$$f^{-1}[V_1 \cap V_2] \quad = \quad f^{-1}[V_1] \;\cap\; f^{-1}[V_2]$$

also belongs to $\mathcal{T}_X$ and hence $V_1 \cap V_2$ belongs to $f_*\mathcal{T}_X$.■

(*ii*) Suppose that $\mathcal{U}$ is a topology on X such that $f : (X, \mathcal{U}) \to (Y, \mathcal{T}_Y)$ is continuous. Then $V \in \mathcal{T}_Y$ implies that $f^{-1}[V] \in \mathcal{U}$, and therefore $\mathcal{U}$ containes $f^*\mathcal{T}_Y$. To complete the proof, it will suffice to show that the latter defines a topology on X. Clearly $\emptyset$ and X belong to $f^*\mathcal{T}_Y$ because they are the inverse images of the open sets $\emptyset$ and Y respectively. Suppose now that $V_\alpha \in f^*\mathcal{T}_Y$ for all $\alpha \in A$. Then for each α we have $V_\alpha = f^{-1}[U_\alpha]$ for some $U_\alpha \in \mathcal{T}_Y$, and since $\mathcal{T}_Y$ is a topology for Y we know that

$$f^{-1}\left[\bigcup_{\alpha \in A} U_\alpha\right] \quad = \quad \bigcup_{\alpha \in A} f^{-1}[U_\alpha] \quad = \quad \bigcup_{\alpha \in A} V_\alpha$$

also belongs to $f^*\mathcal{T}_Y$. Similarly, if V_1 and V_2 belong to $f^*\mathcal{T}_Y$ and $V_i = f^{-1}[U_i]$ for $U_i \in \mathcal{T}_X$ and $i = 1, 2$, then

$$f^{-1}[V_1 \cap V_2] \quad = \quad f^{-1}[V_1] \;\cap\; f^{-1}[V_2] \quad = \quad U_1 \;\cap\; U_2$$

also belongs to $\mathcal{T}_Y$ and hence $V_1 \cap V_2$ belongs to $f^*\mathcal{T}_Y$.■