Supplement to Chapter 6 of Sutherland,

Introduction to Metric and Topological Spaces (Second Edition)

Avoiding plausible but false conclusions

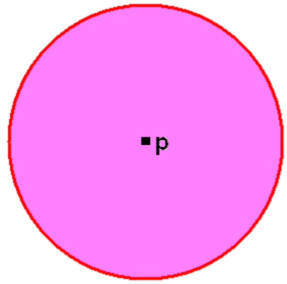
From an intuitive viewpoint, many results about metric spaces probably seem very obvious to at least some readers. For example, the validity of Proposition 5.26 on page 51 of Sutherland (a finite union of bounded subsets is bounded) quickly becomes apparent if one draws a picture showing two bounded subsets of the plane. Clearly one can always draw a circle with a large enough radius to enclose the two bounded subsets, no matter how far apart they might be located. In this case, the proof of the proposition in Sutherland is an abstract logical justification of normal geometrical intuition; furthermore, the proof verifies that the conclusion is extremely general in nature and not limited to examples which can be represented very simply on a sheet of paper.

Although geometric and logical intuition are often extremely for studying and grasping the basic concepts and results for metric spaces, it is frequently misleading, and for this reason **it is always necessary to give logical arguments which justify intuitive conclusions, no matter how obvious they may seem.** Frequently a situation is far less simple than it appears at first.

> [E]vidence ... may seem to point very straight to one thing, but if you shift your own point of view a little, you may find it pointing in an equally uncompromising manner to something entirely different.
>
> A. C. Doyle (1859 – 1930), ***Sherlock Holmes*** **:** ***The Boscombe Valley Mystery***

> For every problem, there exists a simple and elegant solution which is absolutely wrong. **[**Note: A "solution" of this sort is rarely unique.**]**
>
> J. B. Wagoner (1942 –)

One simple example of this type is given in Exercise 6.12 on page 73 of Sutherland. Given a point $\mathbf{p}$ in a metric space $\mathbf{X}$ and a positive constant $\mathbf{r}$, the open ball or disk $\mathbf{B_r(p)}$ of radius $\mathbf{r}$ centered at $\mathbf{p}$ is the set of all $\mathbf{x}$ in $\mathbf{X}$ such that the distance from $\mathbf{p}$ to $\mathbf{x}$ is less than $\mathbf{r}$. If $\mathbf{X}$ is $\mathbb{R}^{\mathbf{n}}$, then it is not difficult to check that the boundary of $\mathbf{B_r(p)}$ is the set of all points $\mathbf{y}$ whose distance from $\mathbf{x}$ is equal to $\mathbf{r}$. In the drawing on below, the open disk is colored in pink and its boundary (which is the set of all points $\mathbf{y}$ with $\mathbf{d(y, p) = r}$) in red.



One part of the exercise is to generalize half of this statement to arbitrary metric spaces:

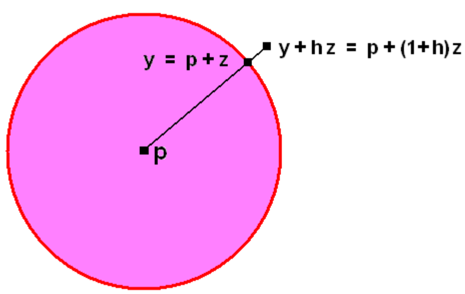
The closure of $\mathbf{B_r(p)}$ **is contained in** the **closed disk** of all points $\mathbf{y}$ such that $\mathbf{d(y, p) \le r}$.

The remainder of the exercise is to prove that, in general, ***the closure of the open disk is not necessarily equal to the closed disk***. In particular, this shows that some basic notions for metric spaces do not necessarily have all the good properties which hold in important and relatively simple cases.

Here is a related example involving the notions of closure and interior for a subset of a metric space $\mathbf{X}$. We start with the following statement:

The open subset $\mathbf{B_r(p)}$ in the coordinate plane is equal to the interior of its closure.

Here is a proof: Since $\mathbf{B_r(p)}$ is open and contained in its closure, it follows that this set is contained in the interior of its own closure. To complete the proof, first note that the closure of $\mathbf{B_r(p)}$ is contained in the set of points $\mathbf{y}$ such that $\mathbf{d(y, p) \le r}$, so it will suffice to prove that if $\mathbf{d(y, p) = r}$ then $\mathbf{y}$ cannot lie in the interior of the closure. Assume now that $\mathbf{y}$ does satisfy this equation, and let $\mathbf{z = y - p}$. If $\mathbf{V}$ is an open set containing $\mathbf{y}$ and $\delta > 0$ is such that $\mathbf{B_\delta(y)}$ is contained in $\mathbf{V}$, then it follows that $\mathbf{y + hz}$ lies in $\mathbf{V}$ if $\mathbf{0 < h < \delta/|z| = \delta/r}$. By construction, $\mathbf{y + hz = p + (1+h)z}$; therefore, if we let $\mathbf{V}$ denote the interior of the closure of $\mathbf{B_r(p)}$ and assume that $\mathbf{y}$ lies in $\mathbf{V}$, then it will follow that $\mathbf{y + hz}$ also lies in $\mathbf{V}$.

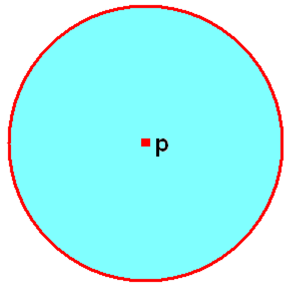


On the other hand, the drawing above suggests that $\mathbf{y + hz = p + (1+h)z}$ does not lie in the closure of $\mathbf{B_r(p)}$, and we can prove this as follows: If a point $\mathbf{x}$ lies in this closure then as before we know that $\mathbf{d(p, x) \le r}$, but we have

$$\mathbf{d(p, y + hz) = (1+h)|z| = (1+h)r > r}$$

so that $\mathbf{y + hz}$ cannot lie in the closure of the open disk, and *a fortiori* it cannot lie in the interior of this closure.

The relative simplicity of this example may suggest that ***every open subset might be equal to the interior of its closure***. However, this is definitely false. Perhaps the simplest counterexamples in the coordinate plane are given by letting **W = B_r(p) – { p }**; for these open sets, the interior of the closure of **W** is equal to **B_r(p).** In the drawing below, the boundary points of **W** are depicted in red.

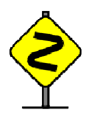


An open set which is equal to the interior of this closure is said to be a regular open set. Many basic properties of such sets are developed in Exercise 22 on page 92 of the following book:

J. Dugundji. ***Topology.*** Allyn and Bacon, Boston, MA, 1965.

The online site **http://planetmath.org/encyclopedia/RegularOpenSet.html** also contains further information on this topic.

NOTATION. In mathematical writings, material which can easily be misinterpreted, leading to hopelessly false conclusions, is often marked by "dangerous bend" symbolism like the following:



http://en.wikipedia.org/wiki/File:Knuth%27s_dangerous_bend_symbol.svg

Some additional examples

Closures of countable unions. If $\{A_k\}$ is an infinite sequence of closed sets (indexed by, say, the nonnegative integers), then it is tempting to think that the closure construction satisfies the identity

$$\overline{\cup A_k} = \cup \overline{A_k}$$

especially since the analogous equation is valid for a finite sequence of closed sets (by Exercise 6.14 on page 74 of Sutherland). However, it is easy to construct examples where this is not the case. Perhaps the simplest counterexample involves the real line, where we take A_k to be the set **{1/k}**; in this case each one point set $\{A_k\}$ is closed in the real line, but the union of the closures (the left hand set) also contains the point **{0}**. It is not difficult to prove that the second set is contained in the first (***Hint:*** Use Proposition 6.11(b) on page 63 of Sutherland).

Note. Exercise 7 on page 101 of Munkres, ***Topology***, has a fallacious proof of the purported identity displayed above, and the goal of the exercise is to find the mistake.

Finally, here is a similar problem which is left to the reader:

Question. Suppose that $\mathbf{f : X \longrightarrow Y}$ is continuous, and let **A** be a subset of **X** such that **p** is a limit point of **A.** Is **f(p)** a limit point of **f[A]?** Prove this or give a counterexample**.**

Some references

Since topological spaces are so general in nature**,** it is eventually necessary to place additional conditions on them in order to obtain nontrivial theorems**.** There is an extremely wide range of standard properties in point set topology**,** and one basic theme in the subject is to determine whether or not one given property **P** logically implies a second property **Q.** Obviously a direct proof is the usual method for showing that **P** implies **Q,** while showing that **P** does not imply **Q** is generally done by constructing an example of a topological space which satisfies **Q** but not **P.** The following classic text is the standard source of examples for proving that one property **Q** is not a consequence of another property **P** (or some combination of properties)**.**

L. A. Steen and J. A. Seebach Jr**.** ***Counterexamples in Topology*** (Reprint of the 1970 Ed.)**.** Dover Publications, Mineloa, NY**,** 1995**.**

There is also a searchable online database which contains a great deal of information about this topic**:**

http://austinmohr.com/home/?page_id=146