Second Supplement to Chapter 7 of Sutherland,

Introduction to Metric and Topological Spaces (Second Edition)

Exercise 7.1 in Sutherland asks for an enumeration of the different topologies on the standard sets $\{0,1\}$ and $\{0,1,2\}$ with two and three elements respectively, and the solution to the problem described on page 2 of the online file

`http://fdslive.oup.com/www.oup.com/booksites/pdf/uk/companion/9780199563081/S.7.pdf`

shows that the numbers of topologies are **4** for $\{0,1\}$ and **29** for $\{0,1,2\}$. However, in some respects there are redundancies in these enumerations; for example, the two Sierpiński Space topologies on $X = \{0,1\}$ given by

$$\mathcal{S}_0 \;=\; \{\,\emptyset, \{0\}, X\,\}\ , \qquad \mathcal{S}_1 \;=\; \{\,\emptyset, \{1\}, X\,\}$$

are essentially the same, the only difference being that the roles of 0 and 1 are interchanged. Formally, we can summarize this relationship by stating that if σ is the permutation of $X = \{0,1\}$ which interchanges 0 and 1, then the mapping

$$\sigma : (X, \mathcal{S}_0) \;\longrightarrow\; (X, \mathcal{S}_1)$$

is a homeomorphism. We can generalize this as follows:

Definition. Let $X = \{0, \cdots, n-1\}$ be a standard set with n elements, let $\mathcal{T}$ be a topology on X, and suppose that σ is a permutation of X. Define $\sigma_*\mathcal{T}$ to be the family of all subsets of X which have the form $\sigma[V]$ where $V \in \mathcal{T}$. — Before proceeding to the main definition, we shall give a few simple properties of this construction:

LEMMA. *(a) If σ is the identity permutaton, then $\sigma_*\mathcal{T} = \mathcal{T}$. Furthermore, if σ and τ are two permutations, then $\sigma_*(\tau_*\mathcal{T}) = (\sigma \circ \tau)_*\mathcal{T}$.*

(b) The map σ defines a homeomorphism from $(X, \mathcal{T})$ to $(X, \sigma_*\mathcal{T})$.

Examples. Even if σ is not the identity, then $\mathcal{T}$ and $\sigma_*\mathcal{T}$ might still be equal. For example, if $\mathcal{T}$ is either the discrete or indiscrete topology, then we can check directly that $\sigma_*\mathcal{T} = \mathcal{T}$.

Proof. (a) We first prove that $\sigma_*\mathcal{T} = \mathcal{T}$ if σ is the identity. If $V \in \mathcal{T}$ then $\sigma[V] = V$ implies that $V \in \sigma_*\mathcal{T}$, and conversely if $W \in \sigma_*\mathcal{T}$ then $W = \sigma[V]$ for some $V \in \mathcal{T}$, and since σ is the identity we have $\sigma[V] = V$, so that $W = V \in \mathcal{T}$. Therefore we have shown that each of $\mathcal{T}$ amd $V \in \sigma_*\mathcal{T}$ is contained in the other.

The identity $\sigma_*(\tau_*\mathcal{T}) = (\sigma \circ \tau)_*\mathcal{T}$ follows directly from the identity $\sigma \circ \tau[V] = \sigma_*\,[\,\tau_*[V]\,]$.■

(b) The mapping σ is 1–1 onto because it is a permutatoin, so we need to verify that σ is both continuous and open when viewed as a map from $(X, \mathcal{T})$ to $(X, \sigma_*\mathcal{T})$.

To see that σ is open, note that if V is open with respect to $\mathcal{T}$ then $\sigma[V]$ is open with respect to $\sigma_*\mathcal{T})$. To see that σ is continuous, start with a subset W which is open with respect to $\sigma_*\mathcal{T})$. By the definition of the latter, we know that $W = \sigma[V]$ where V is open with respect to $\mathcal{T}$. Since σ is 1–1 onto, it follows that the inverse image $\sigma^{-1}[W]$ is equal to V, and since the latter is open it follows that σ is also continuous. Combining the observations of this paragraph, we see that σ is a homeomorphism from $(X, \mathcal{T})$ to $(X, \sigma_*\mathcal{T})$.■

Definition. Given X as above and two topologies $\mathcal{T}$ and $\mathcal{T}'$ on X (which may coincide), write $\mathcal{T} \sim \mathcal{T}'$ if and only if $\mathcal{T}' = \sigma_* \mathcal{T}$. The first part of the lemma implies that this relation is reflexive and transitive, and the relation is symmetric because $\mathcal{T}' = \sigma_* \mathcal{T}$ implies $\mathcal{T} = \tau_* \mathcal{T}'$ where $\tau = \sigma^{-1}$.

The equivalence classes of topologies on X with respect to this relation are called **homeomorphism types** (or **topological types**) with underlying set X. In these terms, we are interested in the following question:

> *If $X = \{0, \cdots, n-1\}$ is the standard set with n elements, describe the homeomorphism types of topological spaces with underlying set X.*

If $n = 1$ then there is only one topology (which is both a discrete and an indiscrete topology!), so there is a unique homeomorphism type. If $n = 2$, then as noted above, the solution to Exercise 7.1(a) in Sutherland implies that there are **4** topologies and they lie in **3** equivalence classes (the classes of the discrete, indiscrete and Sierpiński topologies).

Homeomorphism types for $X = \{0, 1, 2\}$

Our main objective in this document is to determine the number of distinct homeomorphism types when $n = 2$; by the solution to 7.1(b) in Sutherland there are 29 topologies, and we need to describe the various equivalence classes.

One way of sorting topologies on the finite set $X = \{0, \cdots, n-1\}$ is to consider the number $N_k(\mathcal{T})$ of subsets U in the topology $\mathcal{T}$ such that U contains exactly k points, where $1 \leq k \leq n-1$ (we know that $N_0 = N_n = 1$ by the definition of a topological structure). It follows immediately that if $\mathcal{T} \sim \mathcal{T}'$ then for each k we have $N_k(\mathcal{T}) = N_k(\mathcal{T}')$, so the sequence of integers N_k provide one simple way to begin the study of homeomorphism types of topologies on X.

The following observations will be extremely helpful in analyzing homeomorphism types:

DUALITY PRINCIPLE. *If $X = \{0, \cdots, n-1\}$ and $\mathcal{T}$ is a topology on X, then the set $\mathcal{T}^*$ of all complements of subsets in $\mathcal{T}$ is also a topology on X.*

This follows because $\mathcal{T}^*$ contains $\emptyset = X - X$ and $X = X - \emptyset$, and the usual DeMorgan laws relating union, intersection and complementation

$$(X - A) \cup (X - B) \quad = \quad X - (A \cap B) \ , \qquad (X - A) \cap (X - B) \quad = \quad X - (A \cup B)$$

show that $\mathcal{T}^*$ is closed under taking unions and intersections (since X is finite, there are only finitely many subsets in a topology for X, so every intersection is actually a finite intersection).■

COROLLARY 1. *In the setting of the Duality Principle, suppose that a is a nonnegative integer and $1 \leq j \leq n-1$. Then the number of topologies on X with $N_j = a$ is equal to the number of topologies on X with $N_{n-j} = a$, and likewise for the number of homeomorphism types of topologies.*

Note that for some choices of n and a the number of homeomorphism types will be trivial; for example, this is true if

$$a \quad > \quad \binom{n}{j} \ .$$

Proof. If $\mathcal{T}$ is a topology on X then by definition we have $N_j(\mathcal{T}^*) = N_{n-j}(\mathcal{T})$ because the sets in $\mathcal{T}^*$ are complements of the sets in $\mathcal{T}$. Since $\mathcal{T}^{**} = \mathcal{T}$, every topology $\mathcal{T}$ is complementry to some unique topology $\mathcal{U}$, and hence for each topology with $N_j(\mathcal{T}) = a$ there is a uniquely associated topology $\mathcal{U}$ with $N_{n-j}(\mathcal{U}) = a$. This proves the first part of the conclusion.

To prove the statement about homeomorphism types, note that if $\mathcal{U}$ is complementary to $\mathcal{T}$ and σ is a permutation of X, then $\sigma_*\mathcal{U}$ is complementary to $\sigma_*\mathcal{T}$, and hence the complementary topology construction sends equivalent topologies (with respect to the relation $\sim$) into equivalent topologies. Therefore complementation sends an equivalence class of topologies into an equivalence class of topologies.■

COROLLARY 2. *In the setting of the preceding result, suppose that we are given a sequence of nonnegative integers $a_1, \cdots, a_{n-1}$. Then the number of topologies on X with $N_j = a_j$ for all j such that $1 \leq j \leq n-1$ is equal to the number of topologies on X with $N_j = a_{n-j}$ for all j such that $1 \leq j \leq n-1$, and likewise for the number of homeomorphism types of topologies.*

Proof. This is just a compound version of the preceding corollary, with one hypothesis and one conclusion for each j such that $1 \leq j \leq n-1$.■

If we specialize to the case $n = 3$, then the approach of the preceding paragraph indicates that we should split the analysis of topological structures into cases depending upon the pair of integers N_1 and N_2, which is essentially the approach in the online document cited above. Since $n = 3$, we must have $0 \leq N_1, N_2 \leq 3$, but not all pairs (N_1, N_2) of this type can be realized; for example, $(N_1, N_2) = (3, 0)$ is impossible because if there are three subsets with one element, then the conditions for a topology imply that there must also be three subsets with two elements (take the union of two subsets which contain exactly one point). The Duality Principle and its corollaries essentially cut the amount of work needed to analyze these cases in half, for they imply that if we can find the number of homeomorphisms types with a given (N_1, N_2) where $N_1 \geq N_2$, then we can retrieve the numbers with a given (N_1, N_2) where $N_1 \leq N_2$ because the numbers of types with $(N_1, N_2) = (p, q)$ and $(N_1, N_2) = (q, p)$ are equal.

We are now ready to look at the various cases.

Cases $(N_1, N_2) = (3, k)$, where $0 \leq k \leq 3$. There is only one topology; namely the discrete topology for which $N_2 = 3$. To see that $N_2 < 3$ is impossible, note that if $N_1 = 3$ then a topology $\mathcal{T}$ must contain every one point set, and since $\mathcal{T}$ is closed under unions it must also contain every other set.

Case $(N_1, N_2) = (0, 0)$. In this case the only possible topology is the indiscrete topology.

Case $(N_1, N_2) = (1, 0)$. If $\{p\}$ is the unique open subset in the topology containing one element, then there is a permutation sending the original topology into one for which $\{0\}$ is the unique open subset. In this case there is only one open subset aside from $\emptyset$ and X, so there is only one homeomorphism class of topologies in this case, and there are three topologies in this homeomorphism class.

Case $(N_1, N_2) = (1, 1)$. As in the preceding case, the original topology is equivalent to one for which $\{0\}$ is the unique open subset with one element, so we have reduced this case to considering homeomorphism types of topologies such that $\{0\}$ is the unique open subset with one element. There are three possibilities for the unique open subset with two elements, and one can check that each of the possibilities $\{0, 1\}$, $\{0, 2\}$ and $\{1, 2\}$ can be realized as the unique two point subset for some topology. The first two topologies define the same homeomorphism type because the permutation σ which interchanges 1 and 2 will send one to the other.

On the other hand, the third topology is not in the same homeomorphism class. — To see this, first observe that if σ sends the third topology to either of the others, then since $\{0\}$ is the unique subset with one element we must have $\sigma(0) = 0$, which means that either σ is the identity or else σ interchanges 1 and 2. But σ sends the third topology to itself, and therefore it follows that the third topology cannot lie in the same homeomorphism class as the other two. To summarize, there are two homeomorphism classes of topologies in this case; the class in this paragraph consists of three topologies, and the class in the preceding paragraph consists of six topologies.

Case $(N_1, N_2) = (2, 0)$. This case is impossible, for if the topology contains two one point subsets then it contains their union, which must be a two point subset.

Case $(N_1, N_2) = (2, 1)$. As before, the original topology has the same homeomorphism type as a topology whose one point open subsets are $\{0\}$ and $\{1\}$, so we may reduce everything to finding the homeomorphism types of topologies for which these are the one point subsets. As in the preceding discussion, we know that $\{0, 1\}$ is open, and hence this must be the unique open subset with exactly two points. Therefore there is only one homeomorphism class of topologies in this case, and there are three topologies in this homeomorphism class.

Case $(N_1, N_2) = (2, 2)$. Once again, we may reduce everything to finding the homeomorphism types of topologies for which the one point open subsets are $\{0\}$ and $\{1\}$. As in the preceding case, one of the two point open subsets must be $\{0, 1\}$, and hence the other such set must be either $\{0, 2\}$ or $\{1, 2\}$. One can check that each of these families is a topology for X, and the permutation σ with interchanges 0 and 1 sends each into the other. Therefore there is only one homeomorphism class of topologies in this case, and there are six topologies in this homeomorphism class.

By the consequences of the Duality Principle, the preceding yields the numbers of homeomorphism types, and we can summarize everything in the following 4×4 matrix in which $c_{p,q}$ denotes the number of homeomorphism types such that $(N_1, N_2) = (p, q)$, where $0 \le p, q \le 3$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

If we add up the entries of this matrix, we see that there are **9** homeomorphism types of topologies on $X = \{0, 1, 2\}$.

For purposes of comparison, here is the corresponding matrix whose entries are the numbers of topologies such that $(N_1, N_2) = (p, q)$:

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 3 & 9 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Notice that if we add the entries in this matrix, the sum is 29, which is the number of distinct topologies on X.