Supplement to Chapter 8 of Sutherland,

Introduction to Metric and Topological Spaces (Second Edition)

The middle section of this chapter (*Homeomorphisms*, pp. 84 – 85) describes the condition for mathematical equivalence of topological spaces. One formulation of this condition is that there is a **1 – 1** correspondence between the two spaces — say **X** and **X′** — such that a subset **U** of **X** is open (in **X**) if and only if the corresponding set of points **U′** of **Y** is open (in **X′**). The material in this document explains the relation between this concept and the colloquial phrase

Topology is rubber sheet geometry

which often appears in popularizations of the subject.

Homeomorphisms and equivalence of topological spaces

Since the concept of equivalence in the preceding paragraph clearly differs from the standard definition of homeomorphism in topology, we shall first prove that these notions are equivalent.

PROPOSITION. *Let* **X** *and* **Y** *be topological spaces, let* $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ *be a* **1 – 1** *correspondence of the underlying sets, and let* **g** *be its inverse function. Then the following are equivalent:*

1. *For each subset* **U** *of* **X**, **U** *is open in* **X** *if and only if* **f[U]** *is open in* **Y**.
2. *Both* **f** *and its inverse function* **g** *are continuous.*

Proof. *The first statement implies the second.* We can split the first statement into two parts; namely, if **U** is open in **X** then **f[U]** is open in **Y**, and if **V** is open in **Y** then **g[V]** is open in **X.** The first part is equivalent to the continuity of **g**, and the second is equivalent to the continuity of **f.**

The second statement implies the first. This follows from essentially the same considerations: If **V** is an open subset of **Y**, then **f** is continuous if and only if the inverse image of **V** in **X** is open, and this inverse image is equal to **g[V].** Likewise, if **U** is an open subset of **X**, then **g** is continuous if and only if the inverse image of **U** in **Y** is open, and this inverse image is equal to **f[U].■**

Congruence, similarity and metric spaces

The first step in relating homeomorphisms to "rubber sheet geometry" is to explain why some standard geometric transformations are examples of homeomorphisms.

Probably the most straightforward concept of equivalence for geometric objects in Euclidean space is the notion of **congruence**. If two geometric figures **A** and **B** (formally, ***subsets***) in Euclidean space are congruent, then there is a **1 – 1** correspondence $\mathbf{f}: \mathbf{A} \to \mathbf{B}$ such that for all points **x** and **y** in **A** we have the distance identity $\mathbf{d_2(x, y) = d_2(f(x), f(y))}$, where $\mathbf{d_2}$ is the Euclidean distance defined on pages 40 – 41 of Sutherland.

Another fundamental notion for equivalence for geometric figures is ***similarity***; if two geometric figures **A** and **B** are similar, then there is a **1 – 1** correspondence $\mathbf{f: A \longrightarrow B}$ and a positive constant **c** such that for all points **x** and **y** in **A** we have the distance identity $\mathbf{d_2(f(x), f(y)) = c \cdot d_2(x, y)}$. The constant **c** is called the ***ratio of similitude***; by the preceding discussion, congruent figures are similar such that the ratio of similitude is **1.**

These notions generalize directly to metric spaces, in which a ***similarity transformation*** from one (subset of a) metric space **A** to another (subset of a) metric space **B** is a **1 – 1** correspondence $\mathbf{f: A \longrightarrow B}$ such that for all points **x** and **y** in **A** we have the distance identity $\mathbf{d_B(f(x), f(y)) = c \cdot d_A(x, y)}$ for some positive constant **c,** and an ***isometry*** is a defined to be a similarity transformation for which $\mathbf{c = 1}$. We note that the spaces **A** and **B** may be the same. Further information on the logical relationship(s) between the standard geometric concept of congruence and the purely metric notion of isometry appears in the following online document:

http://math.ucr.edu/~res/math133/metgeom.pdf

The following result shows that similarity transformations are examples of homeomorphisms.

PROPOSITION. *If* $\mathbf{f: A \longrightarrow B}$ *is a similarity transformation, then* **f** *is a homeomorphism*.

The proof will actually establish a stronger result; namely, both **f** and its inverse function **g** are uniformly continuous (see page 135 of Sutherland).

Proof. Let **c** be the ratio of similitude for **f**. Then for all **a** in **A** and all $\varepsilon > 0$ we know that $\mathbf{d_A(x, a) < \varepsilon/c}$ implies

$$\mathbf{d_B(f(x), f(a)) = c \cdot d_A(x, a) < c \cdot (\varepsilon/c) = \varepsilon}$$

so that **f** is continuous at **a** for all **a** in **A** (and in fact f is uniformly continuous).

Now let **g** be the inverse function to **f**; since **f** is a **1 – 1** correspondence it follows that

$$\mathbf{d_B(g(u), g(v)) = c^{-1} \cdot d_A(u, v)}$$

for all **u** and **v** in **B,** which means that the inverse function **g** is a similarity transformation with ratio of similitude $\mathbf{c^{-1}}$. Therefore **g** is (uniformly) continuous by the reasoning of the preceding paragraph.■

Affine transformations

Frequently it is useful to consider relationships between geometric figures that are less restrictive than congruence or similarity. One example is ***affine equivalence***: Two subsets **A** and **B** of $\mathbb{R}^n$ are **affine equivalent** if there is a mapping from $\mathbb{R}^n$ to itself of the form

$$\mathbf{F(p) = T(p) + k}$$

where **T** is an invertible linear transformation of $\mathbb{R}^n$ and **k** is some vector in $\mathbb{R}^n$, such that **F** maps **A** onto **B**; composite mappings of this sort are called ***affine transformations***. Results from **http://math.ucr.edu/~res/math133/metgeom.pdf** imply that congruent or similar figures are affine equivalent (see the results stated on pages **1, 3** and **12,** and the comments at the top of the latter page).

Many basic results in elementary geometry involve conditions under which geometrical figures are congruent or similar, and it is not difficult to state results of this type for affine equivalence. For our purposes it seems more useful to discuss congruence, similarity and affine equivalence for a simple but important class of objects.

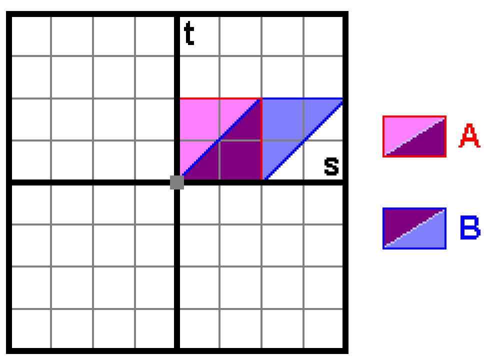
EXAMPLE. If $\mathbf{p}$ and $\mathbf{q}$ are real numbers such that $\mathbf{0 < q \le p}$, let $\mathbf{A(p, q)}$ denote the solid rectangular region $\mathbf{[0, p] \times [0, q]}$ in the coordinate plane $\mathbb{R}^2$. For these examples it is fairly simple exercise to prove the following:

1. For all $\mathbf{(p, q)}$ and $\mathbf{(r, s)}$ as above, $\mathbf{A(p, q)}$ is congruent to $\mathbf{A(r, s)}$ if and only if $\mathbf{(p, q)}$ and $\mathbf{(r, s)}$ are equal.
2. For all $\mathbf{(p, q)}$ and $\mathbf{(r, s)}$ as above, $\mathbf{A(p, q)}$ is similar to $\mathbf{A(r, s)}$ if and only if $\mathbf{p/r}$ and $\mathbf{q/s}$ are equal (equivalently, if and only if $\mathbf{p/q = r/s}$).
3. For ***all*** $\mathbf{(p, q)}$ and $\mathbf{(r, s)}$ as above, $\mathbf{A(p, q)}$ is affine equivalent to $\mathbf{A(r, s)}$.

A detailed verification is given in Appendix A.

Here is a slightly less elementary example, which is in part motivated by considerations from multivariable calculus.

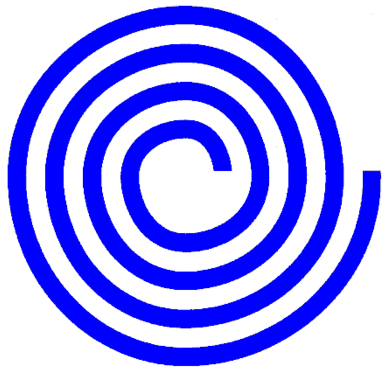
Shear transformation. The solid parallelogram – shaped region $\mathbf{B}$ defined by the equations $\mathbf{0 \le y \le 2}$ and $\mathbf{y \le x \le y + 2}$ is affine equivalent to the solid rectangle $\mathbf{A = [0,2] \times [0,2]}$, where the affine transformation $\mathbf{F : A \longrightarrow B}$ is defined by $\mathbf{F(s, t) = (s + t, t)}$.



In multiple integration this affine equivalence is sometimes useful because the mapping $\mathbf{F}$ and the standard Change of Variables formula for double integrals reduce the computation of an integral over $\mathbf{B}$ to the computation of an integral over the rectangular region $\mathbf{A}$.

More general transformations

Of course, there are also many other transformations that are useful in multivariable calculus. For example, the polar coordinate mapping $\mathbf{(x, y) = (r\cos\theta, r\sin\theta) = \mathcal{P}(r, \theta)}$ provides the standard method for reducing the computation of a double integral over the spiral – shaped region defined by the inequalities $\theta + \frac{1}{2} \le \mathbf{r} \le \theta + \mathbf{1}$ and $\mathbf{0 \le \theta \le 8\pi}$ (see the drawing below) to the computation of a double integral over $\mathbf{[\frac{1}{2}, 1] \times [0, 8\pi]}$; in particular, this reduction can be applied to show that the area of the spiral region is equal to **$11\pi^2 + 3\pi$.** Appendix B contains a proof that *the polar coordinate transformation defines a homeomorphism from this rectangular region onto the given spiral region*.



(***Adapted from*** **http://2.bp.blogspot.com/-IAXyyMXIU2g/TbphOaKwbXI/AAAAAAAAAEw/zifLpCucuYg/s1600/spiral1.gif**)

The preceding examples suggest that ***homeomorphisms are general, abstract versions of transformations related to classical geometry and the standard change of variables mappings in multivariable calculus.***

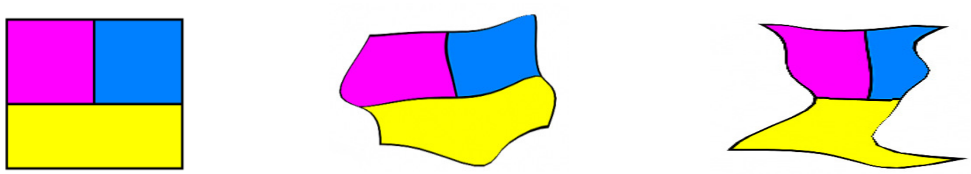
Geometric properties of homeomorphisms

In applications of the Change of Variables Theorem for multiple integrals to specific examples, it is usually necessary to get some geometric insight into the behavior of the transformation defining the change of variables. More generally, one major theme in topology is to provide qualitative information about homeomorphisms. The following quotation summarizes some of the most important general statements on this problem:

> In topology, the movements we are allowed [homeomorphisms] might be called ***elastic motions*** [emphasis added]. We imagine that our figures are made of perfectly elastic rubber and, in moving a figure, we can stretch [or compress], twist, pull and bend it at pleasure. … However, we must be careful that distinct points in a figure remain distinct; we are not allowed to force two [or more] different points to coalesce into just one point.
>
> B. H. Arnold, ***Intuitive Concepts in Elementary Topology*** (Reprint of the 1962 Edition; Dover Publications, Mineola, NY, 2011), pp. 23 – 24.

In the planar case, the stretching, shrinking, bending and twisting operations can be carried out by manipulating an ideally elastic rubber sheet, and *this is why topology is sometimes known as rubber sheet geometry*.

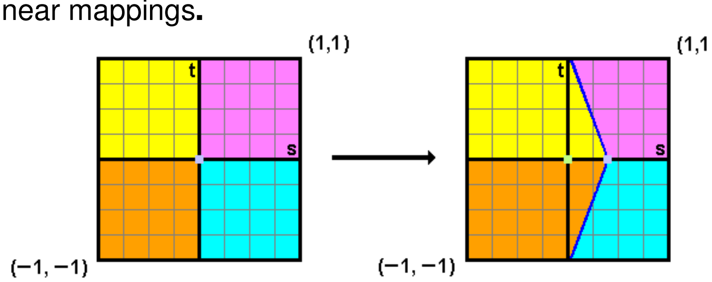
There are numerous interactive software applications which allow a user to experiment fairly easily with stretching, shrinking and bending figures. The images below give two results of such operations on the figure at the left; Appendix C contains specific information on the software used here.



In order to justify this rigorously, it is necessary to describe a homeomorphism mapping the set on the left to one of the sets on the right in explicit analytic terms. For the given examples, the computer program and user input can be analyzed to obtain very accurate numerical information on the behavior of the homeomorphism which seems to exist. However, ***for theoretical purposes we need a complete, explicit description of a homeomorphism in terms of analytic formulas.*** We shall illustrate this process with a nontrivial example that is fairly simple intuitively and can also be described by relatively simple formulas.

The drawings below are meant to suggest there is a homeomorphism of the solid square region $\mathbf{X} = [-1, 1] \times [-1, 1]$ with the following geometric properties:

1. The center point $(0, 0)$ is mapped to some point $(a, 0)$ with $0 < a < 1$.
2. Each boundary point is sent to itself (the restriction to the boundary is inclusion).
3. The t – axis is sent to the broken line with two segments, one joining $(0, -1)$ to $(a, 0)$ and the other joining $(a, 0)$ to $(0, 1)$; in the right hand square, this image is the blue broken line.
4. The square regions formed by intersecting $\mathbf{X}$ with the four closed quadrants are mapped to trapezoids as indicated, such that the horizontal slices $t = c$ in the square regions on the left are sent to the horizontal slices $t = c$ in the trapezoidal regions on the right by linear mappings.



The given conditions indicate that map bends the t – axis is bent in the middle, it stretches the two subregions to the left of that axis, and it shrinks the two subregions to the right of that axis. An explicit equation for the image of the t – axis is $s = a - a \cdot |t|$, and using this it is fairly straightforward to derive the following explicit formula for the mapping under consideration:

$$F(s, t) = (a - a \cdot |t| + s(1 - a + a \cdot |t|), t) \qquad \text{if } s \geq 0$$

$$F(s, t) = (a - a \cdot |t| + s(1 + a - a \cdot |t|), t) \qquad \text{if } s \leq 0$$

One can check directly that both formulas yield the same value on the set of points where both $s \geq 0$ and $s \leq 0$ (so that $s = 0$). In order to complete the discussion, we now need to prove rigorously that this map is a homeomorphism, and it will suffice to define a function which can be checked to be an inverse to F. We can find this function by solving $(x, y) = F(s, t)$ for s and t in terms of x and y as follows: First of all, in each case we have $t = y$. This reduces everything to solving the equation $x = a - a \cdot |y| + s(1 - a + a \cdot |y|)$ for s when $x \geq a - a \cdot |y|$ and solving the equation $x = a - a \cdot |y| + s(1 + a - a \cdot |y|)$ when $x \leq a - a \cdot |y|$. Since $0 < a < 1$ and $|y| \leq 1$ the coefficients of s in these equations satisfy

$$0 < 1 - a + a \cdot |y| < 1 < 1 + a - a \cdot |y|$$

and hence they can be solved uniquely for s in terms of x and y. Finally, each of the formulas yields the same value for s in the overlapping case where $x = a - a \cdot |y|$, so the inverse mapping is in fact well – defined.

In conclusion, two side issues

Homeomorphisms and motions. As noted on pages 111 – 112 of the previously cited book by Arnold, there is a significant drawback with the informal description of homeomorphisms as elastic motions.

> Unfortunately, the term "elastic motion" carries with it some … intuitive connotations. Chief among these … is the idea that "motion" from one place to another necessarily entails some sort of path, or route, along which this motion takes place. … No path, or route, is needed for a transformation [given by a homeomorphism].

One way of formalizing the concept of motion is to consider a family of homeomorphisms which varies over time, and this can be done by adding a real variable corresponding to time. For example, we can define a ***motion*** (or ***deformation***) of a subset $\mathbf{A}$ in a topological space $\mathbf{X}$ to be a continuous function $\mathbf{M}$ from $\mathbf{A} \times [0, 1]$ to $\mathbf{X}$ such that $\mathbf{M(a, 0) = a}$ and for each $\mathbf{z}$ in $[0, 1]$ the restricted mapping $\mathbf{M_z(a) = M(a, t)}$ sends the subset $\mathbf{A}$ homeomorphically onto its image $\mathbf{M_z[A]}$. If $\mathbf{X}$ is a metric space, we can also define a more restrictive concept of ***rigid motion*** in which we insist that each of the mappings $\mathbf{M_z}$ be distance preserving.

The homeomorphisms described on the preceding page can be extended to motions very easily; all we need to do is replace $\mathbf{a}$ by $\mathbf{b_z = (1 - z)\, a}$, where $\mathbf{z}$ lies in $[0, 1]$. If $\mathbf{z = 0}$ then we obtain the identity map, and if $\mathbf{z = 1}$ we recover the original homeomorphism.

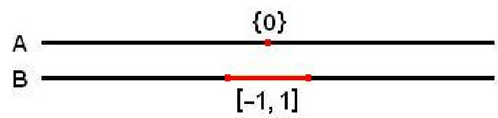
It is important to know that <u>*not every homeomorphism of a topological space*</u> $\mathbf{X}$ <u>*to itself can be extended to a motion in the sense described above*</u>. For example, let $\mathbf{X} = \mathbb{R}$ and consider the homeomorphism $\mathbf{S}$ sending $\mathbf{x}$ to $\mathbf{-x}$. Suppose that there is some motion $\mathbf{M}$ as above such that $\mathbf{M_1 = S}$. Consider now the continuous function $\mathbf{g(t) = M(1, t) - M(-1, t)}$. By our hypotheses we know that $\mathbf{g(0) = 2}$ and $\mathbf{g(1) = -2}$. Therefore the Intermediate Value Theorem implies that $\mathbf{g(c) = 0}$ for some $\mathbf{c}$ between $\mathbf{0}$ and $\mathbf{1}$, which means that $\mathbf{M(1, c) = M(-1, c)}$, contradicting our assumption that $\mathbf{M_c}$ was $\mathbf{1-1}$. This contradiction implies that no motion with the given properties can exist.

<u>*Final remark.*</u> Popular descriptions of topology often include whimsical statements about a doughnut and coffee cup being topologically equivalent; the underlying idea is that a sufficiently pliable doughnut can be reshaped to the form of a coffee cup by creating a dimple in the latter and progressively enlarging it, while shrinking the hole into a handle. Here is a link to an animation which illustrates this point particularly well:

http://en.wikipedia.org/wiki/File:Mug_and_Torus_morph.gif

Ambiently homeomorphic subsets. If two geometrical figures $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{R}^{\mathbf{n}}$ are affine equivalent via some affine transformation $\mathbf{F}$ (in particular, if they are congruent or similar), then $\mathbf{F}$ is a homeomorphism from $\mathbb{R}^{\mathbf{n}}$ to itself which sends $\mathbf{A}$ to $\mathbf{B}$. More generally, we shall say that two subsets $\mathbf{A}$ and $\mathbf{B}$ of a topological space $\mathbf{X}$ are ***ambiently homeomorphic*** subspaces if there is a homeomorphism $\mathbf{F}$ from $\mathbf{X}$ to itself such that $\mathbf{F[A] = B}$. It follows immediately that if $\mathbf{A}$ and $\mathbf{B}$ are ambiently homeomorphic, then the map $\mathbf{h : A \longrightarrow B}$ defined by $\mathbf{h(a) = F(a)}$ defines a homeomorphism from $\mathbf{A}$ to $\mathbf{B}$; in fact, the inverse function to $\mathbf{h}$ is given by the inverse function to $\mathbf{F}$.

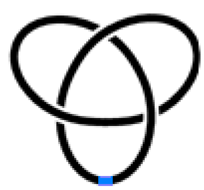
In general, the converse to the preceding statement is false; *homeomorphic subsets of a space are not necessarily ambiently homeomorphic*. For example, let $\mathbf{X} = \mathbb{R}$ and set $\mathbf{A}$ and $\mathbf{B}$ equal to $\mathbb{R} - \{0\}$ and $\mathbb{R} - [0, 1]$ respectively. Equivalently, $\mathbf{A}$ is the union of the disjoint intervals $(-\infty, 0)$ and $(0, +\infty)$ while $\mathbf{B}$ is the union of the disjoint intervals $(-\infty, -1)$ and $(1, +\infty)$. An explicit homeomorphism from $\mathbf{A}$ to $\mathbf{B}$ is given by $\mathbf{g(x)} = \mathbf{x} + \mathrm{sgn}(\mathbf{x})$, where $\mathrm{sgn}(\mathbf{x})$ is $\mathbf{1}$ if $\mathbf{x}$ is positive and $-\mathbf{1}$ if $\mathbf{x}$ is negative; the inverse function is $\mathbf{y} - \mathrm{sgn}(\mathbf{y})$.



(The complementary subspaces are in **red**.)

Now if $\mathbf{F}$ were a homeomorphism from $\mathbb{R}$ to itself sending $\mathbf{A}$ to $\mathbf{B}$, then $\mathbf{F}$ would also map the complement of $\mathbf{A}$ to the complement of $\mathbf{B}$ in a $\mathbf{1} - \mathbf{1}$ manner. But these respective complements are $\{0\}$ and $[0, 1]$ respectively; since the first complement is finite and the second is infinite, there is no $\mathbf{1} - \mathbf{1}$ correspondence between them, and thus by ***reductio ad absurdum*** it follows that a homeomorphism with the desired properties cannot exist.

Here is one fundamental, and extensively studied, problem in topology which involves ambient homeomorphisms: ***Given two regular smooth simple closed curves in*** $\mathbb{R}^3$, ***when are they ambiently homeomorphic***? One of the simplest cases in which they are not ambiently homeomorphic involves a standardly embedded (planar) circle and the following curve, which is essentially obtained by cutting a circle at some point, tying a knot, and gluing the two cut ends back together again; the particular example shown below is called a ***trefoil*** (TREFF – foil) ***knot***, and the blue spot is meant to suggest the point at which one cuts the curve and glues it back together.



(***Source:*** **http://chesterfieldpagans.files.wordpress.com/2010/06/trefoil.png?w=500**)

A proof of the assertion about the circle and the trefoil knot is beyond the scope of this course, but here is a detailed introductory reference, which is written at the advanced undergraduate level:

> **R. H. Crowell and R. H. Fox.** *Introduction to Knot Theory* (Reprint of the 1963 Edition). Dover Publications, Mineola, NY, 2008.

Appendix A: *Congruence, similarity and affinity for rectangles*

We shall now prove the proposition on page 3, which gives the conditions under which two standard rectangular regions **A(p, q)** and **A(r, s)** are congruent, similar or affine equivalent. For the purposes of this document, two planar figures are congruent if and only if there is an isometry from one onto the other (see **http://math.ucr.edu/~res/math133/metgeom.pdf** for further information; in particular, the proposition on page 3 implies that isometries and similarities of

$\mathbb{R}^n$ preserve angle measurements, and as noted on pages 2 – 3 of this document every isometry or symmetry of geometric figures in $\mathbb{R}^n$ extends to an isometry or symmetry on all of $\mathbb{R}^n$). The results of the online document also show that every isometry of $\mathbb{R}^n$ has the form $\mathbf{F(p) = T(p) + k}$ where $\mathbf{T}$ is an invertible linear isometry of $\mathbb{R}^n$ and $\mathbf{k}$ is some vector in $\mathbb{R}^n$; likewise, a similarity transformation on $\mathbb{R}^n$ has the form $\mathbf{F(p) = cT(p) + k}$ where $\mathbf{c}$ is a positive constant, $\mathbf{T}$ is an invertible linear isometry of $\mathbb{R}^n$ and $\mathbf{k}$ is some vector in $\mathbb{R}^n$. In particular, these results imply that congruent or similar figures in $\mathbb{R}^n$ are affine equivalent as defined earlier in this document.

For the sake of clarity we shall repeat the statements we want to prove. Recall that if $\mathbf{u}$ and $\mathbf{v}$ are real numbers such that $\mathbf{0 < v \leq u}$, then the solid rectangular region $\mathbf{[0, p] \times [0, q]}$ in the coordinate plane $\mathbb{R}^2$ is denoted by $\mathbf{A(u, v)}$.

1. For all **(p, q)** and **(r, s)** as above, **A(p, q)** is congruent to **A(r, s)** if and only if **(p, q)** and **(r, s)** are equal.
2. For all **(p, q)** and **(r, s)** as above, **A(p, q)** is similar to **A(r, s)** if and only if $\mathbf{p/r}$ and $\mathbf{q/s}$ are equal (equivalently, if and only if $\mathbf{r/p = s/q}$).
3. For ***all*** **(p, q)** and **(r, s)** as above, **A(p, q)** is affine equivalent to **A(r, s)**.

We shall begin by verifying the "***only if***" parts of the three statements. For the first statement, there is nothing to prove because the condition implies that $\mathbf{A(p, q) = A(r, s)}$. For the second statement, let $\mathbf{c = r/p = s/q}$; in this case the invertible linear transformation $\mathbf{T(p) = cp}$ maps **A(p, q)** onto **A(r, s)**. Finally, for the third statement the invertible linear transformation $\mathbf{T(x, y) = (rx/p, sy/q)}$ maps **A(p, q)** onto **A(r, s)**.

To complete the discussion, we need to prove the "***if***" parts of the three statements. For the third statement, there is nothing to prove because the conclusion places no restrictions on the pairs **(p, q)** and **(r, s)**. In the first two cases, we know that there is a **1 – 1** correspondence $\mathbf{f : A(p, q) \to A(r, s)}$ such that for all points $\mathbf{x}$ and $\mathbf{y}$ in **A(p, q)** we have the distance identity $\mathbf{d_2(f(x), f(y)) = c \cdot d_2(x, y)}$ for some positive constant $\mathbf{c}$, with $\mathbf{c = 1}$ in the first case. As noted earlier, results from **http://math.ucr.edu/~res/math133/metgeom.pdf** show that there is a similarity transformation $\mathbf{F}$ of $\mathbb{R}^2$ which extends $\mathbf{f}$, and this transformation is an isometry of $\mathbb{R}^2$ in the first case. Since similarity transformations are affine transformations, we can apply Corollary 7 from **http://math.ucr.edu/~res/math145A-2013/affine+convex.pdf** to obtain some information about the behavior of $\mathbf{f}$ and $\mathbf{F}$.

LEMMA A1. *Let* **F** *a similarity transformation of* $\mathbb{R}^2$ *sending* **A(p, q)** *onto* **A(r, s).** *Then* **F** *maps the vertices of* **A(p, q)** *onto* **A(r, s).**

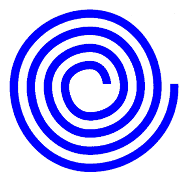
Conclusions of the proofs. Assume first that **A(p, q)** is similar to **A(r, s)** with ratio of similitude **c.** In general, for every **(u, v)** such that $\mathbf{0 < v \leq u}$, the distances between the various pairs of distinct vertices for **A(u, v)** are given as follows:

1. For **{(0, 0), (0, v)}** and **{(u, 0), (u, v)}**, the distance is equal to **v.**
2. For **{(0, 0), (u, 0)}** and **{(0, v), (u, v)}**, the distance is equal to **u.**
3. For **{(0, 0), (u, v)}** and **{(u, 0), (0, v)}**, the distance is equal to $\mathbf{sqrt(u^2 + v^2)}$.

These distances are in the order $0 < v \le u < \mathrm{sqrt}(u^2 + v^2)$. Of course, if $(u, v) = (r, s)$ then the ordered list of differences is $0 < s \le r < \mathrm{sqrt}(r^2 + s^2)$; however, since we are given that $A(p, q)$ is similar to $A(r, s)$ with ratio of similitude c, the ordered list of distances for the rectangular region $A(r, s)$ is also equal to $0 < cq \le cp < c \cdot \mathrm{sqrt}(p^2 + q^2)$, and since these two ordered lists must be identical it follows that $s = cq$ and $r = cp$. Therefore we have $r/p = c = s/q$. This proves the "***if***" part of the second statement. The proof for the "***if***" part of the first statement is essentially the same; this is merely the special case for which the ratio of similitude c is equal to 1.■

Appendix B: *Proof that polar coordinates determine a homeomorphism*

We shall prove that the polar coordinate map sends the rectangular region $[\frac{1}{2}, 1] \times [0, 8\pi]$ homeomorphically onto the spiral ribbon described on pages 3 – 4 of this document; we recall that the defining inequalities are $\theta + \frac{1}{2} \le r \le \theta + 1$ and $0 \le \theta \le 8\pi$. In the drawing below, the spiral region is colored in blue.



The spiral band defined by the preceding inequalities is the image of the rectangular region in the $r\theta$ – plane under the composite $\mathcal{P} \circ F$, where $\mathcal{P}$ is the polar – to – rectangular coordinate mapping $\mathcal{P}(r, \theta) = (r\cos\theta, r\sin\theta)$ and $F(r, \theta) = (r + \theta, \theta)$. Since we want to show this composite is $1 - 1$ on the given rectangular region, it will be helpful to recall the conditions under which polar coordinates map two points in the $r\theta$ – plane to the same point in the xy – plane:

<u>POLAR COORDINATE AMBIGUITY.</u> *Let* r *and* s *be nonnegative, and suppose that* θ *and* φ *are such that* $\mathcal{P}(r, \theta) = \mathcal{P}(s, \varphi)$. *Then* $r = s$ *and one of the following holds*:

1. *We have* $r = s = 0$, *and* $\mathcal{P}(r, \theta) = \mathcal{P}(s, \varphi)$ *for all* θ *and* φ.
2. *We have* $r = s > 0$, *and* $\mathcal{P}(r, \theta) = \mathcal{P}(s, \varphi)$ *if and only if* $\theta - \varphi = 2k\pi$ *for some integer* k.

In particular, if A is a subset of the $r\theta$ – plane on which $r > 0$ and such that $\mathcal{P}$ is not $1 - 1$, then there is a pair of elements in A of the form (r, θ) and $(r, \theta + 2k\pi)$ for some ***nonzero*** integer k.

Let B be the rectangle described above, and let $A = F[B]$. For a fixed $C > 0$, we need to find all θ such that (C, θ) belongs to A; the defining inequalities for A imply that

$$C - 1 < \theta < C - \tfrac{1}{2}$$

and therefore A does not contain a pair of points of the form (C, θ) and $(C, \theta + 2k\pi)$ for some nonzero integer k. Thus the preceding discussion shows that the restriction of $\mathcal{P}$ to A is $1 - 1$, and since F is $1 - 1$ it follows that the restriction of $\mathcal{P} \circ F$ to B is also $1 - 1$. A consequence of the "Inverse Function Theorem for Compact Metric Spaces" (specifically, Corollary 13.27 on page 136) then implies that $\mathcal{P} \circ \mathbf{F}$ maps the rectangular region **B** homeomorphically onto the spiral band.

Appendix C: *Interactive software for deforming images*

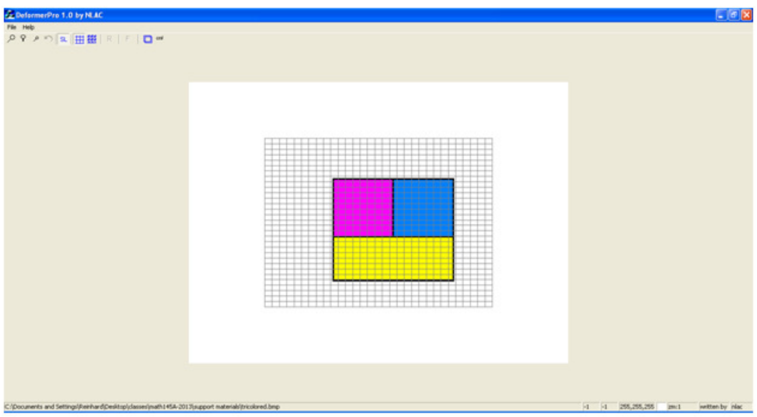
The images at the bottom of page 4 were created using the software application **DeformerPro 1.0,** which can be legally downloaded free of charge from the following site:

http://deformer-pro.en.uptodown.com/download

Unlike many sites for downloading free software, this site does not download or install additional unwanted material, and there are not even any annoying prompts that require negative input from the user. The download is in a compressed format, and the following free and versatile unzipping program is highly recommended for this and many other downloads:

http://7-zip.org/

Note on using the program: When a picture is loaded into **DeformerPro 1.0** for processing, a rectangular or triangular grid is superimposed over the picture (see the screen shot below) and one deforms the picture by pointing, clicking and dragging the mouse over the grid.



The program allows deformations that are not **1 — 1,** and in order to ensure that the maps in question are homeomorphisms ***it is necessary to take care so that none of the vertical, horizontal or diagonal lines meet each other*** at any points other than those where they meet when the grid first appears on the screen. This is the case with the left hand picture below but **not** with the right hand picture (notice that parts of the image overlap others).