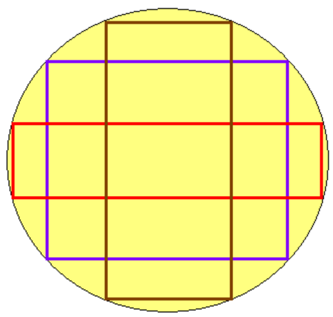
Supplement to Chapter 10 of Sutherland,

Introduction to Metric and Topological Spaces (Second Edition)

Nonrectangular open sets in product spaces

The following example illustrates the insertion in boldface type near the bottom of page 101 in Surtherland; namely, ***open sets in a Cartesian product are generally unions of*** **very large** ***families of rectangular open subsets.***

If we are given an open disk in the plane of the form $\mathbf{x^2 + y^2 < r^2}$ then it is fairly easy to describe this set as a union of rectangular open sets; for example, if we are given a point **(u, v)** in the (open) first quadrant of its boundary circle and we let **W(u, v)** be the rectangular region **(– u, u) × (– v, v),** then the disk is the union of the sets **W(u, v).**



In fact, it is possible to express the open disk as a union of a countably infinite sequence of rectangular open sets (see Munkres, *Topology* (Second Edition), pages 190 – 191 and 194). However, we claim that ***it is not possible to express the open disk as a*** **finite** ***union of rectangular open sets.*** This probably looks obvious from the drawing, but we really have to give a rigorous proof to establish this and thus we shall do so here.

The first step in proving the assertion in the preceding paragraph is to derive the following fact about boundaries of unions of subsets **A, B** in a metric or topological space **X:**

$$\partial(A \cup B) \;=\; \overline{A \cup B} \cap \overline{X - (A \cup B)} \;=\; (\overline{A} \cup \overline{B}) \cap \overline{X - (A \cup B)} \;=$$

$$\left(\overline{A} \cap \overline{X - (A \cup B)}\right) \cup \left(\overline{B} \cap \overline{X - (A \cup B)}\right) \text{ is contained in}$$

$$(\overline{A} \cap \overline{X - A}) \cup (\overline{B} \cap \overline{X - B}) \;=\; \partial A \;\cup\; \partial B.$$

> [***Proof.*** The first equality is just Exercise 6.23(b) on page 75 of Sutherland, the second follows from Proposition 6.13 on page 64 of Sutherland, the third equality follows from the distributivity of unions over intersections, the set – theoretic containment follows from the obvious inclusions $\mathbf{X - (A \cup B) \subset X - A, X - B}$ and Proposition 6.11(b) on page 63 of Sutherland, and the final equality — just like the first — follows from Exercise 6.23(b) on page 75 of Sutherland.]

If we apply the displayed containment relation repeatedly, we see that ***the boundary of a finite union of rectangular open subsets is contained in the union of the boundaries of the individual rectangular open subsets.*** In particular, it follows that ***the boundary of such a finite union is contained in a finite union of vertical and horizontal lines.***

To see that an open disk of the form $\mathbf{x^2 + y^2 < r^2}$ cannot be a finite union of rectangles, note first that its boundary set is the circle with equation $\mathbf{x^2 + y^2 = r^2}$, and in particular the boundary must be infinite. Next, note that the intersection of this boundary with an arbitrary horizontal or vertical line $\mathbf{x = a}$ or $\mathbf{y = b}$ must contain at most two points. If the disk were a finite union of rectangular open sets, then since the boundary would be contained in a finite union of vertical and horizontal lines it would follow that at least one of these lines would contain infinitely many boundary points of the open disk. This yields a contradiction, the source of which is the assumption that the open disk is a finite union of rectangular open sets. Therefore the disk cannot be a finite union of this sort, proving our assertion.