Second Supplement to Chapter 10 of Sutherland,

Introduction to Metric and Topological Spaces (Second Edition)

If we are given two functions $f : (a, b] \to \mathbb{R}$ and $g : [b, c) \to \mathbb{R}$ such that $f(b) = g(b)$, then it is often useful to define a new continuous function $h : (a, c) \to \mathbb{R}$ such that $h(x) = f(x)$ on $(a, b]$ and $h(x) = g(x)$ on $[b, c)$; the condition $f(b) = g(b)$ ensures that the function is well-defined. Of course, similar conclusions hold if f or g is defined on the closed interval $[a, b]$ or $[b, c]$ respectively, in which case the domain of definition for h can be extended to include a or c respectively. In particular, we can use this principle to define the absolute value function $|x|$ and prove its continuity. Other examples were considered in the portions of this course corresponding to Chapter 6 in Sutherland; for example, if $f : [a, b] \to \mathbb{R}$ is continuous, then f can be extended to all of $\mathbb{R}$ by setting $f(x)$ equal to $f(a)$ if $x \leq a$ and $f(x) = b$ for $x \geq b$. Strictly speaking, the latter involves two applications of the principle in the first sentence of this paragraph; we need to begin by defining the extended function on $(-\infty, b]$ by taking f to be the constant function with value $f(a)$ on $(-\infty, a]$, and we finish by defining the extended function on all of $\mathbb{R}$ by taking f to be the constant function with value $f(b)$ on $[b, +\infty)$.

The purpose of this document is to formulate and prove a similar method for piecing together continuous functions on arbitrary topological spaces.

THEOREM. *Let X be a topological space, let E and F be closed subsets of X, and suppose that $f : E \to Y$ and $g : F \to Y$ are continuous functions into the same topological space Y. If the restrictions $f|E \cap F$ and $g|E \cap F$ are equal, then there is a unique continuous function $h : X \to Y$ such that $h|E = f$ and $h|F = g$.*

Proof. The condition on restrictions implies that there is a unique function of sets $h : X \to Y$ such that $h|E = f$ and $h|F = g$; we need to show that h is continuous if f and g are continuous.

Let $K \subset Y$ be a closed set; we need to prove that $h^{-1}[K]$ is closed in X. Since $X = E \cup F$ we also have

$$h^{-1}[K] \quad = \quad \left(E \cap h^{-1}[K]\right) \ \cup \ \left(F \cap h^{-1}[K]\right) \quad = \quad f^{-1}[K] \ \cup \ g^{-1}[K]$$

where the last equation holds because such that $h|E = f$ and $h|F = g$. Since f and g are continuous, the sets $f^{-1}[K]$ and $g^{-1}[K]$ are closed in E and F respectively, and since E and F are both closed in X it follows that both $f^{-1}[K]$ and $g^{-1}[K]$ are closed in X. Since the union of two closed subsets is closed, the displayed formulas imply that $h^{-1}[K]$ is closed in X and hence h is continuous.■