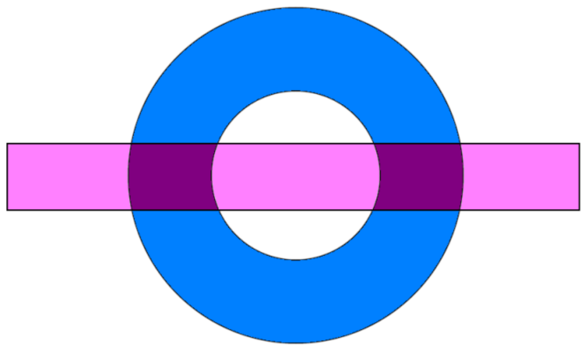
Supplement to Chapter 12 of Sutherland,

Introduction to Metric and Topological Spaces (Second Edition)

Intersections of connected subsets

In contrast to Proposition 12.16 on page 118 of Sutherland, ***the intersection of two connected sets is not necessarily connected.*** The drawing below illustrates this for a particular pair consisting of a closed rectangular region **A** and a closed ring – shaped (annular, pronounced ANN – you – lar) region **B.** The disconnectedness of the intersection (the purple shaded region) is apparent from the drawing, but as usual we need to give a rigorous proof that the intersection of **A** and **B** is not connected.



We assume that the first region is a rectangle centered at the origin with (k, h) as one of its vertices, while the second is the set of all points whose polar coordinates satisfy $a \leq r \leq b$. In order for this construction to produce the desired examples, it will be necessary to assume that $h < a < k$. The first set is connected by Theorem 2.18 on page 119 of Sutherland, while the second is connected because it is the image of the rectangular region $[a, b] \times [0, 2\pi]$ under the standard polar coordinate transformation (and hence is connected by Proposition 12.11 on page 117 of Sutherland).

To prove the intersection is disconnected, first note that it is symmetric with respect to the $y-$axis, so that (x, y) belongs to the set if and only if $(-x, y)$ does. Next, we note that the minimum $x-$coordinate of a point in the right hand piece of the intersection is $\mathrm{sqrt}(a^2 - h^2)$. To see this, observe that we have $|y| \leq h$, $x \geq 0$ and $a^2 \leq x^2 + y^2$ by construction, and these imply that $x \geq \mathrm{sqrt}(a^2 - y^2) \geq \mathrm{sqrt}(a^2 - h^2)$. The value on the right hand side is realized when we take (x, y) to be $(a, 0)$.

By construction $\mathrm{sqrt}(a^2 - h^2)$ is positive, and hence all points in the intersection have nonzero first coordinates. If we now let **U** and **V** be the sets of points in the intersection whose first coordinates are positive and negative respectively, then it follows that **U** and **V** are disjoint open subsets whose union is the entire intersection. They are nonempty because the first contains $(a, 0)$ while the second contains $(-a, 0)$. Therefore it follows that the intersection of the rectangular shaped set and the ring shaped set is disconnected.