Conditions for contiuity of inverse functions

If $f : X \to Y$ is a continuous function which is 1–1 and onto, then there is a set-theoretic inverse to f, but there are also many examples to show that this inverse function need not be continuous. However, there are numerous conditions of a general nature which imply the continuity of inverses. One of the simplest examples involves linear transformations. Basic results in linear algebra imply the following:

Theorem. *If $T : V \to W$ be a linear transformation of finite-dimensional vector spaces which is $1-1$ and onto, then the inverse mapping T^{-1} from W to V is also a linear transformation. In paticular, if V and W are inner product spaces, then T^{-1} is linear and hence T is a homeomorphism (since linear transformations on finite-dimensional inner product spaces are continuous).*

Similarly, Proposition 13.26 in Sutherland shows that if X and Y are compact Hausdorff spaces and $f : X \to Y$ is continuous and 1–1 onto, then f is a homeomorphism (and hence its inverse is continuous). The main purpose of this document is to derive yet another result of this type which comes from multivariable calculus:

Global Inverse Function Theorem. *Let U and V be open subsets in $\mathbb{R}^n$, and let $f : U \to V$ be a $1-1$ onto continuous function which satsifies the following differentiability conditions:*

- (i) *If the coordinate functions for f are given by $y_1, \cdots, y_n$ then each function y_i has continuous partial derivatives.*
- (ii) *The Jacobian determinant*

$$\frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)} = \det\left(\frac{\partial y_i}{\partial x_j}\right)$$

is nonzero at every point of U.

Then f is a homeomorphism.

We shall prove this using the following standard result from multivariable calculus:

Local Inverse Function Theorem. *Let U be an open subset of $\mathbb{R}^n$, let p be a point in U, and let $f : U \to \mathbb{R}^n$ be a continuous function which satsifies the following differentiability conditions:*

- (i) *If the coordinate functions for f are given by $y_1, \cdots, y_n$ then each function y_i has continuous partial derivatives near p.*
- (ii) *The Jacobian determinant*

$$\frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)} = \det\left(\frac{\partial y_i}{\partial x_j}\right)$$

is nonzero at p.

Then f has a local inverse; specifically, there are open neighborhoods V and W of p and $f(p)$ respectively such that $V \subset U$, f induces a $1-1$ onto mapping from V to W, and the inverse mapping $g : W \to V$ is also a function whose coordinates have continuous partial derivatives at every point.

This central result in multivariable calculus can be found in virtually every multivariable calculus textbook, so we shall not try to give a reference. However, we note one important consequence.

Corollary. *In the setting of the Local Inverse Function Theorem, the restricted map $f|V$ sends open subsets of the open set V into open subsets of $\mathbb{R}^n$.*

Note that open subsets of V can be interpreted either with respect to the subspace topology on V or with respect to the usual topology on $\mathbb{R}^n$.

Proof of the Corollary. Let $h : V \to W$ be the map defined by f. Then the theorem implies that h is a homeomorphism and accordingly sends open subsets of V into open subsets of W. Since V and W are both open in $\mathbb{R}^n$, it follows that if V_0 is an open subset of $\mathbb{R}^n$ contained in V, then $h[V_0] = f[V_0]$ is an open subset of $\mathbb{R}^n$.■

Deriving the global theorem from the local theorem

If we apply the local theorem and its corollary, we see that if $p \in U$ then there is some open neighborhood V_p of p such that $V_p \subset U$ and $f|V_p$ is an open mapping.

Suppose now that Ω is an open subset in U. Then we have $\Omega = \cup_p \Omega \cap V_p$ and hence

$$f[\Omega] \quad = \quad f\left[\bigcup_{p\in\Omega} \Omega \cap V_p\right] \quad = \quad \bigcup_{p\in\Omega} f\left[\Omega \cap V_p\right] \ .$$

By the first paragraph of this proof, each of the summands on the right hand side is an open set in $\mathbb{R}^n$, and therefore their union, which is merely $f[\Omega]$, is also an open set in $\mathbb{R}^n$. This means that f is an open mapping. Since we are given that f is continuous and 1–1 onto, the preceding sentence implies that f maps U homeomorphically onto $f[U]$.■

Two generalizations

There are many ways in which this result has been generalized in higher mathematics. We shall mention two which often appear in introductory graduate level courses.

Invariance of Domain. (L. E. J. Brouwer) *Let $f : U \to \mathbb{R}^n$ be a continuous function which is $1-1$ and onto. Then f is an open mapping and hence defines a homeomorphism from U to $f[U]$.*

Versions of this result appear in nearly all introductory books on algebraic topology. The specific formulation given above can be found on page 79 of the following standard textbook:

> **A. Dold,** *Lectures on Algebraic Topology* (Second Edition). Springer-Verlag, New York *etc.*, 1980.

In order to state the second result we need some background. A *Banach space* is a normed vector space which is complete in the associated metric. Given two Banach spaces E and F and a continuous mapping $f : U \to F$ — where U is open in E — then one can formulate a concept of differentiability for a function f and a point $p \in U$ such that the derivative $Df(p)$ lies in the space $\mathcal{L}(E, F)$ of continuous linear transformations from E to F. The space $\mathcal{L}(E, F)$ has a canonical normed vector space structure, so if a function is differentiable everywhere it is meaningful to discuss the continuity of the derivative mapping $Df : U \to \mathcal{L}(E, F)$; details are given in Chapter I of the book by Lang cited below. Note that if E and F are finite-dimensional, then $Df(p)$ corresponds to the usual matrix of partial derivatives for the coordinate functions of f.

We then have the following result:

Local Inverse Function Theorem for Banach Spaces. *Let U be an open subset of the Banach space E, let p be a point in U, and let $f : U \to E$ be a continuous function which satsifies the following differentiability conditions:*

(i) *The derivative of f is defined on a neighborhood U_0 and the associated function $Df : U_0 \to \mathcal{L}(E, E)$ is continuous near p.*

(ii) *The derivative $Df(p)$ is an invertible element of $\mathcal{L}(E, E)$.*

Then f has a local inverse; specifically, there are open neighborhoods V and W of p and $f(p)$ respectively such that $V \subset U$, f induces a $1 - 1$ onto mapping from V to W, and the inverse mapping $g : W \to V$ is also a function with a continuous derivative at every point.■

A standard reference for this result is the following classic graduate textbook:

S. Lang, *Introduction to Differentiable Manifolds.* Springer-Verlag, New York *etc.*, 2002.