4. Review of some Real Analysis

Main issues:

(1) How should we describe the real number system?
(2) Review of infinite sequences
(3) Review of continuous real valued functions.

A sequence of number systems

$\mathbb{N}$ = natural numbers (nonneg. integers)
$\mathbb{Z}$ = (signed) integers
$\mathbb{Q}$ = rational numbers

$\boxed{\mathbb{A} = \text{black-board bold letter}}$

Standard facts about $\mathbb{Q}$.

Distinguished elements 0 and 1
Four arithmetic operations $+, -, \times$ or $\cdot$, $\div$
Satisfying basic rules (axioms)
Ordering relation $>$ or $<$ (opposite)
Satisfying basic rules (axioms)

The real numbers $\mathbb{R}$ contain $\mathbb{Q}$ and satisfy analogs of the basic rules for arithmetic and ordering.

One should be able to approximate elements of $\mathbb{R}$ by elements of $\mathbb{Q}$.

Axioms for the rational numbers

Arithmetic Axioms

For all $q \in \mathbb{Q}$, $q + 0 = 1 \cdot q = q$, $q \cdot 0 = 0$

Commutative Laws $a+b=b+a$ $ab=ba$

Associative Laws $a+(b+c)=(a+b)+c$
$a\cdot(bc) = (ab)\cdot c$

[Values of sums & products do not depend on ordering or bracketry.]

Inverse Laws Given a, there is a unique $-a$ so that $a+(-a)=0$

Given $a \neq 0$, there is a unique $1/a = a^{-1}$ so that $a\cdot(1/a)=1$.

(Negatives and reciprocals)

Distributive Law $a(b+c) = ab+ac$.

Ordering Axioms

$\mathbb{Q}^+$ = positive elements. $1 \in \mathbb{Q}$

$a, b \in \mathbb{Q}^+ \Rightarrow a+b, ab \in \mathbb{Q}$.

Trichotomy One and only one of

$a>0$, $a=0$, $0>a$ is true

Write: $a>0$ if $a \in \mathbb{Q}^+$

Also write $a>b \iff a-b \in \mathbb{Q}^+$.

$\mathbb{Q}$ is the smallest system satisfying these conditions.

Example (decimal fraction approximation)

$r > 0$ real number

For each positive integer N, we can ~~write~~ find a positive integer a and a decimal fraction $\frac{b}{10^N}$, where $0 \le \underset{\substack{\uparrow \\ \text{INTEGER}}}{b} < 10^N$, such that

$$a + \frac{b}{10^N} \le r < a + \frac{b+1}{10^N}$$

$$\left(\text{hence} \quad \left| r - \left(a + \frac{b}{10^N}\right) \right| < \frac{1}{10^N}\right).$$

This is the practical computational description of real numbers which has been the standard since the late 16^{th} century (Simon Stevin).

HOWEVER, it is not good for conceptual or theoretical purposes, and these are central to the course.

One reason is that our understanding of real numbers should not depend upon choosing 10 as a computational base (for example, we might want base 2, 3, 5, 12, 16, 20 or 60).

There is also another issue regarding decimal expansions. If we are given integers a_k such that $0 \leq a_k \leq 9$, we want to know for sure that

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \dots + \frac{a_k}{10^k} + \dots$$

represents a real number, and similarly if we choose another computational base B (in which case $B \leq N-1$).

The search for a good answer to these issues took well over 2000 years, from the discovery that $\sqrt{2}$ is irrational* until the second half of the 19th century.**

The development of calculus during the 17th and 18th centuries made this issue far more urgent than it had been previously.

INFORMAL DESCRIPTION. $\mathbb{R}$ is the largest system which contains $\mathbb{Q}$ and satisfies the "standard facts" listed above.

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* By the Pythagoreans in the 6th century B.C.E.

** Independently done by R. Dedekind and G. Cantor.

There are several logically equivalent ways of describing the real numbers $\mathbb{R}$ as a system extending $\mathbb{Q}$. As in Sutherland, we choose one that requires a minimum amount of extra concepts and verbiage.

<u>Definition</u> $(X, \leq)$ partially ordered set

$$\boxed{\begin{array}{l} a \leq a, \ a \leq b \ \& \ b \leq a \Rightarrow a = b \\ a \leq b \ \& \ b \leq c \Rightarrow a \leq c \end{array}}$$

If $A \subseteq X$, an upper bound for A is an element $u \in X$ such that $a \leq u$ for all $a \in A$; a lower bound is an element $v \in X$ such that $v \leq a$ for all $a \in A$.

<u>Examples</u> (1) $X = \mathbb{Q}$ $A =$ positive rationals $\Rightarrow$ every $q \leq 0$ is a lower bound for A. <u>NO UPPER BOUND</u>

(2) $X = \mathbb{Q}$, $A =$ all x such that $x = 1 - t^2$ for some $t \in \mathbb{Q} \Rightarrow$ every $q \geq 1$ is an upper bound. <u>NO LOWER BOUND</u>

(3) $X = \mathbb{Q}$, $A =$ all x such that $x = \dfrac{1}{1+t^2}$ for some $t \in \mathbb{Q} \Rightarrow$ every q such that

$$\begin{Bmatrix} q \geq 1 \\ q \leq 0 \end{Bmatrix} \text{ is a/an } \begin{Bmatrix} \text{upper} \\ \text{lower} \end{Bmatrix} \text{ bound for } A.$$

Optimal bounds.

Least upper bound (l.u.b., sup[remum]):

$b^* \leq$ every other upper bound.

Greatest lower bound (g.l.b., inf[imum]):

$b^* \geq$ every other lower bound.

Completeness Property of $\mathbb{R}$:
Every nonempty subset $A \subseteq \mathbb{R}$ which has an upper bound also has a least upper bound.

Equivalent version Replace "upper bound" with "lower bound" and "least" with "greatest."

[If B = all $-a$ for $a \in A$, then B has a/an

$\begin{Bmatrix} \text{upper} \\ \text{lower} \end{Bmatrix}$ bound $\Longleftrightarrow$ A has a/an $\begin{Bmatrix} \text{lower} \\ \text{upper} \end{Bmatrix}$ bound].

Proposition If a $\begin{Bmatrix} \text{l.u.b.} \\ \text{g.l.b.} \end{Bmatrix}$ exists, it is unique.

Proof LUB b_1 + b_2 least upper bounds $\Rightarrow$

$b_1 \leq b_2$ + $b_2 \leq b_1$, so $b_1 = b_2$.

GLB b_1 + b_2 greatest lower bounds $\Rightarrow$

$b_1 \geq b_2$ and $b_2 \geq b_1$, so $b_1 = b_2$. ∎