Archimedean Property If $0 < a < b$ in $\mathbb{R}$, then there is a positive integer n such that $na > b$.

Proof Suppose not. Then $na \le b$ for all n, so $X = \{na \mid n \in \mathbb{N}\}$ has an upper bound. By completeness there is some $c = \text{l.u.b.}(X)$. Since $2a \in X$, we have $a < 2a \le c$, and $c - a$ is NOT an upper bound for X. Hence there is some k such that $k \cdot a > c - a$. But then $(k+1)a > c$, contradicting $c = \text{lub}(X)$.

The source of the contradiction is the assumption that $na \le b$ for all n, and hence this must be false; in other words, some $na > b$. ∎

See page 4.13 for a variant of this result

Sutherland, Cor. 4.7 If $a < b$ in $\mathbb{R}$, then there is some rational number q such that $a < q < b$.

Proof. First find a good denominator for the fraction, then find a good numerator.

NEED: Every nonempty subset of $\mathbb{N}$ has a least element (which is unique).

Useful reduction If we can prove the result when $0<a<b$, then the result will be true in all cases

Other options

$a<0<b$: Simply take $q=0$

$a<b<0$: Then $0<-b<-a$. By the special case there is a rational q such that $-b<q<-a$, and this implies $a<-q<b$. Since $-q$ is also rational, this case follows too.

Therefore, in the discussion which follows we shall assume $0<a<b$.

positive integers

First Claim: There is some $n \in \mathbb{N}^+$ such that

$$0 < \frac{1}{n} < b-a.$$

If $0 < b-a < 1$, then by the Archimedean Property there is some n such that $n(b-a) > 1$. Divide both sides of the inequality by n. If $b-a \geq 1$, take $n = 2$. ∎

Find the/a denominator: By the preceding, we can find some $d_1 > 0$ such that $\frac{1}{d_1} < a$ and some $d_2 > 0$ such that $\frac{1}{d_2} < b-a$. Let d be the larger of d_1 & d_2, so that $\frac{1}{d} < a, b-a$.

Numerator:
Choose $c > 0$ to be the least positive integer such that $\frac{c}{d} > a$ (by the Arch. Prop. there are such numbers), and note that $0 \leq \frac{c-1}{d} \leq a < \frac{c}{d}$. But now we have $\frac{c}{d} = \frac{c-1}{d} + \frac{1}{d} < a + (b-a) = b$, so $\frac{c}{d}$ lies between a & b. ∎

We could now go further and justify the standard statements about decimal expansions as on pages 109-120 of the course directory file set-theory-notes.pdf.

once we know the following basic result on infinite series:

<u>Theorem</u> Let $\{x_n\}$ be a sequence of nonnegative terms, and assume that the set of partial sums $S_m = \sum_{n=0}^{m} x_n$ is bounded. Then $\sum_{n=0}^{\infty} x_n$ converges.

Note also that the methods in the cited document work equally well for base B expansions for every integer $B \geq 2$.

The theorem leads to the second main issue in Chapter 4: Infinite sequences and their limits.

$\lim_{n\to\infty} a_n = L \iff$ for each $\varepsilon > 0$ there is some $N \in \mathbb{N}^+$ such that $k \geq N \Rightarrow |a_k - L| < \varepsilon$.

(see Figure 4.1 on p. 22 of Sutherland for a visual description)

The stated theorem on convergent infinite series is a consequence of Thm. 4.16 on p. 24 of Sutherland.

Before proceeding, one observation

<u>CLAIM</u> If $0 \leq b_k \leq 9$ where b_k are integers, then $\displaystyle\sum_{k=1}^{\infty} \frac{b_k}{10^k}$ converges.

<u>Verification</u> Use a comparison test

$\dfrac{b_k}{10^k} \leq \dfrac{9}{10^k}$ and the geometric series formula implies that $\displaystyle\sum_{k=1}^{\infty} \frac{9}{10^k} = 1$, so the partial sums $\displaystyle\sum_{k=1}^{M} \frac{b_k}{10^k}$ have an upper bound. ∎

Of course, a similar argument works for any base $B \geq 2$, with B and $B-1$ replacing 10 and 9 respectively.

Footnote.

The relationship between the concept of an ordered field with the completeness property (p 4.5) and the informal description on p. 4.3 is demonstrated in the file maximality.pdf.

This document is written at a slightly higher level than the course material.