<u>Note</u> The statement of Thm. 4.19 is logically equivalent to the completeness property for the real number system.

READ PAGES 20-25 ON SEQUENCES, AND ALSO READ THE NOTE AFTER ADDITIONAL EXERCISE 1 FOR THIS CHAPTER. (in the file exercises01w14.pdf).

Limits and continuity for functions.

Making limits precise was also a challenge to mathematicians for a long time. The $\delta-\varepsilon$ formulation is due to Weierstrass and was developed in the 19th century.

$\lim_{x \to a} f(x) = b \iff$ for ~~all~~ each $\varepsilon > 0$ there is some $\delta > 0$ such that $0 < |x-a| < \delta \Rightarrow$ $|f(x) - b| < \varepsilon$.

Note that f must be defined on some punctured interval $(a-h, a+h) - \{a\}$, but it may or may not be defined at a; even if it is defined at a, the value of the limit does <u>not</u> depend upon the value $f(a)$.

The file limit-drawings.pdf gives an illustration of $\lim_{x \to a} f(x) = L$ which complements Figures 4.1 and 4.4 on pages 22 and 29 of Sutherland.

It even took mathematicians decades to settle on the def.

"There is no question that the definition of a limit is obscure, and both intuition and experience are generally needed to make the definition more understandable."

Fortunately, many limits can be evaluated fairly simply using a few basic formulas. Many of the latter are summarized in the file limit-laws.pdf; this list does not include l'Hospital's rule because we are interested in objects that do not necessarily have a notion of differentiation. Two comments seem appropriate:

1. The Squeeze Law is discussed further on page 7 of exercises01w14.pdf (specifically in the solution to Exercise 4.14 from Sutherland).

2. The Composition Law and Exercise 4.14 have hypotheses which are very close, but the conclusions are quite different.

The identities in limit-laws.pdf are analogous to Proposition 4.20 on p. 25 of Sutherland (see also pp. 30-32).

Note the following consequence of Lemma 4.25 ($\lim_{x\to a} f(x) = L \iff$ for all $\{a_n\}$ such that $a_n \neq a$, all n, and $\lim_{n\to\infty} a_n = a$, we have $\lim_{x\to a} f(a_n) = L$).

If there are sequences $\{u_n\}$ $\{v_n\}$ which are never equal to a such that $u_n \to a$, $v_n \to a$, $f(u_n) \to L_1$, $f(v_n) \to L_2$ with $L_1 \neq L_2$, then $\lim_{x\to a} f(x)$ DOES NOT EXIST.

Continuity $\lim_{x\to a} f(x) = f(a)$. at $x=a$.

ε-δ statement:
For each $\varepsilon > 0$ there is some $\delta > 0$ such that $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

(Actually a little easier to remember than the definition of a limit!)

Understanding some implications of continuity may be helpful. It's also useful!

Prop. 4.30 If f is continuous at a and $f(a) > 0$, then there is some $\delta > 0$ such that $f > 0$ on $(a-\delta, a+\delta)$.

Proof. Let $\varepsilon = \frac{1}{2} f(a)$, and choose δ accordingly. Then $|x-a| < \delta \Rightarrow$ $|f(x) - f(a)| < \varepsilon$, so that $f(a) - f(x) < \varepsilon =$ $\frac{1}{2} f(a)$, or $f(a) < f(x) + \frac{1}{2} f(a)$, or

$\frac{1}{2} f(a) < f(x)$. Hence $f(x) > 0$. ∎

Propositions 4.31–4.33 on pp. 30-32 of Sutherland give standard results on constructing new continuous functions from old ones by algebra and composite functions.

Finally here are two fundamental results that will be generalized in this course.

Note. Clearly f takes min & max values at different pts of $[a,b]$

BOUNDEDNESS. If $f: [a,b] \to \mathbb{R}$ is continuous, then f is bounded (for some $M > 0$, $f(x) \leq M$ for all x), AND f takes a maximum and a minimum value on $[a,b]$. ∎

(IF f IS NOT CONSTANT)

INTERMEDIATE VALUE PROPERTY. If $f: [a,b] \to \mathbb{R}$ is continuous and $u, v \in [a,b]$ are such that $f(u) < f(v)$, then for each $y \in [f(u), f(v)]$ there is some $c \in [a,b]$ such that $f(c) = y$. ∎

ADDENDUM:

A variant of Sutherland, 4.7

<u>Proposition</u> Let $0 < a < b \leq 1$. Then there is a finite decimal fraction $\frac{c}{10^M}$

such that $a < \frac{c}{10^N} < b$.

<u>Proof</u> We can use the argument as before if we can show there is some N such that

$\frac{1}{10^N} < a$ and $\frac{1}{10^N} < b - a$.

Fortunately, this is easy, for we have

$n < 2^n < 10^n$, so that $\frac{1}{10^n} < \frac{1}{2^n} < \frac{1}{n}$. $\blacksquare$

<u>Corollary</u> If $h > 0$ then there is some $n \in \mathbb{N}^+$ such that $\frac{1}{10^n} < h$. $\blacksquare$

Note that there are analogous results with 10 replaced by an arbitrary number base $B \geq 2$.