5. Metric Spaces

<u>Problem.</u> Find the minimal data needed to establish the results on continuous functions in the previous chapter.

In particular, we want a setting which includes continuous functions defined on various subsets of coordinate n-space $\mathbb{R}^n$.

A few simply stated properties of distance turn out to be sufficient for extending a large class of concepts and results about regions in $\mathbb{R}^n$ and continuous functions.

"a lot of our geometric intuition" carries over to this setting
(Sutherland, ¶1 before Prop 5.2, p.39)

<u>Definition</u> A <u>metric space</u> (X, d) is a pair consisting of a set X (the space) and a function
$$d : X \times X \longrightarrow \mathbb{R}$$
(the <u>metric</u> or <u>distance function</u>)
such that the following hold:

(M1) $d(x,y) \geq 0$, equality $\Leftrightarrow x = y$.

(M2) $d(x,y) = d(y,x)$

(M3) $d(x,z) \leq d(x,y) + d(y,z)$

all x, y, z in X

We often write just X if d is clear in context or more information is unneeded

Results on inner products imply that $\mathbb{R}^n$ is a metric space with $d(x,y) = \|x-y\|$ vector length

$$= \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$ Pythagorean Metric

There are many others in Sutherland.

One family

Normed vector spaces

$(V, |\cdots|)$ V = real vector space

$|v|$ = norm of $v \in V$, $|\cdots| : V \to \mathbb{R}$ s.t.

$$\left.\begin{array}{l} |v| \geq 0, \text{ equality} \Leftrightarrow v = 0 \\ |cv| = |c| \cdot |v| \\ |v_1 + v_2| \leq |v_1| + |v_2| \end{array}\right\} \begin{array}{l} \text{all} \\ v, c \\ v_1, v_2 \end{array}$$

c = scalar

Verifications that $d(x,y) = |x-y|$ defines a metric.

(M1) $d(x,y) = |x-y| \geq 0$.

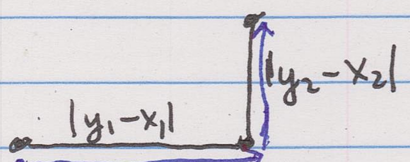
$0 = d(x,y) = |x-y| \iff x-y=0 \iff x=y$.

(M2) $d(y,x) = |y-x| = |(-1)(x-y)| = |-1| \cdot |x-y| = |x-y| = d(x,y)$.

(M3) $d(x,z) = |x-z| = |(x-y)+(y-z)| \leq |x-y| + |y-z| = d(x,y) + d(y,z)$.

Another metric on $\mathbb{R}^n$ comes from the <u>taxicab norm</u>: $|v|_1 = \sum |v_j|$ ← check this is a norm!!

in $\mathbb{R}^2$



$d_1(x,y) = |x-y|$

Geometrically $d_1(x,y)$ in $\mathbb{R}^2$ is the horizontal separation + the vertical separation.

WRITE d_2 FOR THE PYTHAGOREAN METRIC.

<u>Some more examples</u> Many on pp. 40-48 of Sutherland. We shall describe a few that are particularly important.

STANDARD DISCRETE METRICS X = any set

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

(Sutherland, 5.6)
(M1) & (M2) OBVIOUS

Check the Triangle $\leq$: $d(x,z) \leq d(x,y) + d(y,z)$

Case 1 $x = z \Rightarrow d(x,z) = 0$, and we know $d(x,y), d(y,z) \geq 0$, so $d(x,y) + d(y,z) \geq 0$.

Case 2 $x \neq z$ (2A) If $x \neq y$ then

$1 = d(x,z)$ and $d(x,y) + d(y,z) \geq d(x,y) = 1$,

(2B) If $y \neq z$, LHS $\geq d(y,z) = 1$. $= d(x,z)$

IMHO "pathological" (Sutherland, ¶ on pp. 41-42) is too strong, and these spaces DO arise "in nature." However, the rest of the ¶ contains some important points. — It's too soon to try and construct something which is REALLY bizarre (but it definitely can be done!).

"non-standard" might be better

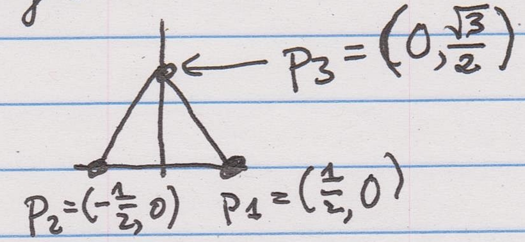
METRIC SUBSPACES (X,d) metric space, $A \subseteq X \Rightarrow$ the subspace metric (d^A or just d) is $d^A = d^X | A \times A$. (Sutherland, 5.8)

Trivia (?) Every metric space "looks like" a subspace of a normed vector space.

What does "looks like" mean?

ISOMETRY = 1-1 onto map $f: X \to Y$ (where (X, d^X) and (Y, d^Y) are metric spaces) such that $d^Y(f(x_1), f(x_2)) = d^X(x_1, x_2)$.

For example, $\{1, 2, 3\}$ with the discrete metric is isometric to the vertices of the following equilateral triangle in $\mathbb{R}^2$:



Reference for the trivia statement: See the Proposition on pp. 50-51 of grad-level-classnotes.pdf (the result is useful in a few situations, but definitely not in many others)

Change of scale metrics If (X, d^X) is a metric space and $k > 0$, so is $(X, k \cdot d^X)$ by Exercise 5.12 in Sutherland.

FUNCTION SPACES. (Modified from Sutherland, 5.13) — These reflect the usefulness of metric spaces in a wide range of contexts — see 5.11 + 5.12 for others)

X = continuous real valued functions on $[0,1]$; if $f : [0,1] \longrightarrow \mathbb{R}$ is continuous let $\|f\| = \max_{t \in [0,1]} |f(x)|$, which exists since $|f|$ attains a maximum value. (f continuous $\Rightarrow$ $|f|$ continuous). Then $\|\cdots\|$ defines a norm on X, so it makes X into a metric space.

(Sutherland, 5.10)

<u>PRODUCT METRICS</u> Given (X, d^X) and (Y, d^Y) define metrics on $X \times Y$ by $\boxed{z = (x,y)}$

$$d_1(z_1, z_2) = d^X(x_1, x_2) + d^Y(y_1, y_2) \quad \begin{pmatrix}\text{Taxicab} \\ \text{metric}\end{pmatrix}$$

$$d_2(z_1, z_2) = \sqrt{d^X(x_1, x_2)^2 + d^Y(y_1, y_2)^2} \quad \begin{pmatrix}\text{Pythagorean} \\ \text{metric}\end{pmatrix}$$

$$d_\infty(z_1, z_2) = \max\{d^X(x_1, x_2), d^Y(y_1, y_2)\} \quad \begin{pmatrix}\text{max} \\ \text{metric}\end{pmatrix}$$

Verifications of metric properties are in the exercises from Chapter 5 of Sutherland.

See <u>product-metrics.pdf</u>, <u>product-metrics$\begin{bmatrix}2\\3\end{bmatrix}$.pdf</u> for a family of metrics d_p, where $1 \leq p \leq \infty$.