Limits and continuity on metric spaces

See Def. 5.3 (p. 40) for continuity. As for limits:

<u>Def.</u> Suppose $a \in X$, metric space, and f is a function from $\{x \in X \mid 0 < d(x,a) < h\}$ to Y, another metric space. Then $\lim_{x \to a} f(x) = L$ means

for each $\varepsilon > 0$ there is some $\delta > 0$ such that
$$0 < d^X(x,a) < \delta \Rightarrow d^Y(f(x), L) < \varepsilon.$$

<u>Expanded Sutherland, p. 48</u> All the results on continuous functions, and the constructions on real valued continuous fcns., carry over to metric spaces.

<u>Example not in Sutherland</u> $f : (X, d^X) \to (Y, d^Y)$ continuous at $a \in X$, and $f(a) \neq b \in Y$. Then there is some $\delta > 0$ s.t. $d(x,a) < \delta \Rightarrow f(x) \neq b$.

<u>PROOF.</u> Let $c = d^Y(f(a), b)$, and choose $\delta > 0$ so that $d(x,a) < \delta \Rightarrow d(f(x), f(a)) < \frac{c}{2}$. Then

$d(x,a) < \delta \Rightarrow f(x) \neq b$, for if $f(x) = b$ then $d(f(x), f(a)) = d(b, f(a)) = c$. ∎

Continuity and product constructions

Given: (X, d^X), (Y, d^Y), $z \in X \times Y$

$z = (x, y)$. d_p metric, $p = 1, 2, \infty$.

THIS TREATMENT DIFFERS FROM SUTHERLAND'S

Coordinate projections $\pi_X : X \times Y \to X$

$\pi_Y : X \times Y \to Y$

$\pi_X(x, y) = x$, $\pi_Y(x, y) = y$

Prop. 5.20 $\pi_X : (X \times Y, d_p) \to (X, d^X)$ and $\pi_Y : (X \times Y, d_p) \to (Y, d^Y)$ are continuous.

New Proof. For each choice of p we have

$d^X(x_1, x_2) \leq d_p(z_1, z_2)$, $d^Y(y_1, y_2) \leq d_p(z_1, z_2)$

so for each $z \in X \times Y$ we can take $\delta = \varepsilon$. ∎

Note These functions are uniformly continuous:

$g : (U, d^U) \to (V, d^V)$ is unif. cont. $\iff$ for each $\varepsilon > 0$ there is some $\delta > 0$ such that, for all $u_1 + u_2$, $d^U(u_1, u_2) < \delta \Rightarrow d^V(g(u_1), g(u_2)) < \varepsilon$.

(Usually the δ for an ε depends on x — think about $f(x) = \frac{1}{x}$ for $0 < x \in \mathbb{R}$).

<u>THEOREM.</u> (W, d^W), (X, d^X), (Y, d^Y) metric spaces, $f:(W,d^W) \longrightarrow (X, d^X)$ and $g:(W,d^W) \longrightarrow (Y,d^Y)$ cont. Define $h: W \longrightarrow X \times Y$ by $h(w) = (f(w), g(w))$. Then $h:(W,d^W) \longrightarrow (X \times Y, d_p)$ is continuous, where $p = 1, 2, \infty$. [Prop. 5.22 is a special case]*

<u>Proof.</u> Let $w \in W$ and $\varepsilon > 0$. Also, let $K(1) = \frac{1}{2}$, $K(2) = \frac{\sqrt{2}}{2}$, $K(\infty) = 1$; in each case, for z_1 & $z_2 \in X \times Y$ we have

$$d^X(x_1,x_2),\ d^Y(y_1,y_2) < \frac{\varepsilon}{K(p)} \Rightarrow d^p(z_1,z_2) < \varepsilon$$

(check this out!). By continuity there exist $\delta_X, \delta_Y > 0$ such that

$$d(t,w) < \delta_X \Rightarrow d(f(t),f(w)) < \frac{\varepsilon}{K(p)}$$
$$d(t,w) < \delta_Y \Rightarrow d(g(t),g(w)) < \frac{\varepsilon}{K(p)}$$

Let $\delta = \min\{\delta_X, \delta_Y\}$. Then by the preceding, $d(t,w) < \delta \Rightarrow \left\{ \begin{array}{l} d(f(t),f(w)) < \frac{\varepsilon}{K(p)} \\ d(g(t),g(w)) < \frac{\varepsilon}{K(p)} \end{array} \right\}$ so that $d_p(h(t),h(w)) < \varepsilon$. ∎

* <u>Note.</u> The diagonal $\Delta: X \longrightarrow X \times X$ is obtained if $f = g =$ idenity map of X.

FOOTNOTE FOR page 5.9

The proof relies heavily on two inequalities:

(1) If $0 \leq u, v < \frac{\varepsilon}{2}$, then $u+v < \varepsilon$

(2) If $0 \leq u, v < \frac{\varepsilon\sqrt{2}}{2}$, then $\sqrt{u^2+v^2} < \varepsilon$.

[So there is a MISTAKE in the fourth line of the proof! Replace $\frac{\varepsilon}{K(p)}$ by $\varepsilon \cdot K(p)$.]

The derivations are straightforward:

(1) The hypothesis implies $0 \leq u+v < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. ∎

(2) The hypothesis implies $0 \leq u^2+v^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2$, so $0 \leq \sqrt{u^2+v^2} < \varepsilon$. ∎

<u>Sutherland, Prop. 5.19</u> Let $f:(X,d^X)\to(U,d^U)$ and $g:(Y,d^Y)\longrightarrow(V,d^V)$ be continuous. Then $f\times g\,(x,y)=(f(x),g(y))$ is continuous.

<u>New proof</u> It helps to draw a picture (a <u>commutative diagram</u>) with all the mappings, such that two composites from the same source to the same target are equal:

$$\begin{tikzcd}
X \arrow[r, "f"] & U \\
X\times Y \arrow[u, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[r, "f\times g"] & U\times V \arrow[u, "\pi_U"'] \arrow[d, "\pi_V"] \\
Y \arrow[r, "g"] & V
\end{tikzcd}$$

Check that the composites are equal!

In the preceding result's setting, let

$F: X\times Y\longrightarrow U$ be $f\circ\pi_X$,

$G: X\times Y\longrightarrow V$ be $g\circ\pi_Y$.

Then $H=f\times g$ by construction.

Now F and G are composites of cont. maps and hence are continuous. By the theorem, this means that $H=f\times g$ is also continuous. ∎

As noted in the ¶ on pp. 49-50 of Sutherland, these results yield slightly simpler (and more generalizable) proofs of statements in Prop. 5.17.

Subsets of metric spaces with special properties

The need to specify various types of subsets in (X, d^X) will arise repeatedly throughout this course (and subsequent ones). We start with a simple example generalizing a property of subsets of $\mathbb{R}$.

$A \subseteq X$ is <u>bounded</u> if for some $x_0 \in X$ we have $d(x_0, a) \leq K$, some $K \geq 0$ (if K works and $K' > K$ then K' also works). Say K is a <u>bound</u>.

<u>Prop. 5.23A</u> If A is bounded, then there is a constant M such that $d(a_1, a_2) \leq M$, all $a_1, a_2 \in A$.

<u>Proof</u> $d(a_1, a_2) \leq d(a_1, x_0) + d(x_0, a_2) \leq 2K$. ∎

The <u>diameter</u> of a bounded set A is the least upper bound of the set of distances $d(a_1, a_2)$ if $A \neq \emptyset$; also, $\operatorname{diam} \emptyset = 0$.

<u>Prop. 5.20</u> A union of two bounded sets is bounded ($\Rightarrow$ same for finitely many, by induction).

<u>Proof.</u> Say $d(x_i, a_i) \le K_i$ if $a_i \in A_i$, where $i = 1$ or 2. Then $d(x_1, a_2) \le d(x_1, x_2) + d(x_2, a_2) \le d(x_1, x_2) + K_2$, so $a \in A_1 \cup A_2 \Rightarrow d(x_1, a) \le$ larger of K_1 and $d(x_1, x_2) + K_2$. ∎

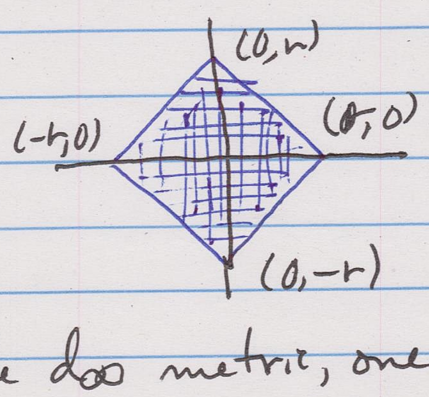
<u>Open neighborhoods of radius r</u>

$$N_r(x) = \{y \in X \mid d(x,y) < r\}.$$

In $\mathbb{R}$ these are open intervals centered at x

In $\mathbb{R}^2$ these are open disks centered at x (with the Pythagorean metric)

In $\mathbb{R}^2$ with the taxicab metric, these are squares whose edges make 45° angles with the coordinate axes:



In $\mathbb{R}^2$ with the d_∞ metric, one gets squares whose edges are parallel to the axes