Basic properties of the family $\mathcal{U}_X$ of open subsets in X.

(1) $\emptyset$ and X are open

Nothing to prove $\quad N_1(x) \subseteq X$ always

(2) A union (possibly infinite!) of open sets is open.

Say U_α open, $\alpha \in \Lambda$ $x \in \cup U_\alpha \Rightarrow x \in U_{\alpha_0}$ some α_0 and hence $N_r(x) \subseteq U_{\alpha_0}$ for some r. But $U_{\alpha_0} \subseteq \cup U_\alpha \Rightarrow N_r(x) \subseteq \cup U_\alpha$ too. ∎

(3) An intersection of two ($\Rightarrow$ finitely many) open sets is open. BY INDUCTION

Say $x \in U_1 \cap U_2$. Choose $s, t > 0$ so $N_s(x) \subseteq U_1$ & $N_t(x) \subseteq U_2$. If $r = \min\{s, t\}$, then $N_r(x) \subseteq U_1 \cap U_2$. ∎

<u>Corollary</u> In a discrete metric, <u>every</u> subset is open.

<u>Proof.</u> $\{x\} = N_{\frac{1}{2}}(x) \Rightarrow \{x\}$ open all $x \in X$.

But $A \subseteq X \Rightarrow A = \bigcup\limits_{a \in A} \{a\}$, so A is open. ∎

Continuity and open subsets

FUNDAMENTAL RESULT $f:(X,d^X)\to(Y,d^Y)$ is continuous $\Leftrightarrow$ for each U open in Y, the inverse image $f^{-1}[U]$ is open in X.

($\Rightarrow$) Suppose f is continuous.
Let $x\in f^{-1}[U]$, so that $f(x)\in U$.
Take $\varepsilon>0$ so $N^Y_\varepsilon(f(x))\subseteq U$. By continuity there is some $\delta_x>0$ so that if $v\in N^X_{\delta_x}(x)$ then $f(v)\subseteq N^Y_\varepsilon(f(x))$, and thus we have

$$N^X_{\delta_x}(x)\subseteq f^{-1}\left[N^Y_\varepsilon(f(x))\right]\subseteq f^{-1}[U].$$

Hence $f^{-1}[U]$ is open in X.

($\Leftarrow$) Suppoce that inverse images of open subsets are open.
x is arbitrary
Let $y\in U$ with $y=f(x)$ and $U=N^Y_\varepsilon(y)$.
Since $f^{-1}[U]$ is open, there is some $\delta>0$ so that $N_\delta(x)\subseteq f^{-1}[U]$. Then $d(v,x)<\delta\Rightarrow v\in N_\delta(x)\Rightarrow f(v)\in U=N^Y_\varepsilon(f(x))$, so that $d(f(v),f(x))<\varepsilon$. Hence f is continuous at $x\in X$. ∎

Finally, some warnings.

1. An infinite intersection of open sets is not necessarily open. Take $U_n \subseteq \mathbb{R}$ with $U_n = (-\frac{1}{n}, \frac{1}{n})$, so $\bigcap_n U_n = \{0\}$. The latter does not contain any subsets of the form $N_r(0) = (-r, r)$ for $r > 0$ because $\frac{r}{2} \notin \{0\}$.

2. As the course progresses we often suppress d^X from (X, d^X) and simply talk about X as a metric space. However, properties of a subset like openness or boundedness depend heavily on the underlying metric. See Sutherland, Example 5.36.

3. If $f: (X, d^X) \to (Y, d^Y)$ is continuous, then U open in X does not imply $f[U]$ is open in Y and vice versa.

   A. $X = Y = \mathbb{R}$, $f =$ constant map w/ value $\{0\}$. Then U open nonempty $\Rightarrow f[U] = \{0\}$, which we have seen is not an open subset of $\mathbb{R}$.

   B. Here is a map f such that U open in $X \Rightarrow f[U]$ is open in Y, but f is not continuous:

Let $X = (\mathbb{R}, d^X)$ $\quad d^X$ = usual metric
$\quad Y = (\mathbb{R}, d^Y)$ $\quad d^Y$ = discrete metric

$$f : X \longrightarrow Y \qquad f(x) = x.$$

Then if U is d^X open, $U = f[U]$ is also d^Y open because all subsets are d^Y open. However, $\{0\}$ is d^Y open but not d^X open, and hence f is not continuous.

Definition If U open in $X \Rightarrow f[U]$ open in Y, we say that f is an open mapping.

Examples $\pi_X : (X \times Y, d_\infty) \longrightarrow X$
$\pi_Y : (X \times Y, d_\infty) \longrightarrow Y$ are open.

This will not be needed for the midterm exam.

We only check the first; the second is similar.
$z = (x, y) \in X \times Y$, $N^\infty_\varepsilon(z) = d_\infty$ neighborhood.

$\pi_X [N^\infty_\varepsilon(z)] = N_\varepsilon(x)$ because
$$N^\infty_\varepsilon(z) = N_\varepsilon(x) \times N_\varepsilon(y).$$

If $U \subseteq X \times Y$ is open and $N^\infty_{\delta(z)}(z) \subseteq U$, then
$$U = \bigcup_z N^\infty_{\delta(z)}(z) \text{ and hence}$$

see the Additional Exercises

$$\pi_X[U] = \pi_X\left[\cup N^{\infty}_{\delta(z)}(z)\right] =$$

$$\pi_X\left[\cup N_{\delta(z)}(\pi_X(z)) \times N_{\delta(z)}(\pi_Y(z))\right] =$$

$\bigcup_z N_{\delta(z)}(\pi_X(z))$, which is a union of open sets and hence is open in X. ∎

Examples with no proof.

$f: \mathbb{C} \to \mathbb{C}$ complex polynomial function (non constant) is open.

(shown in courses on complex variables)

We should note that the corresponding result for real polynomials is false!

The simplest example is the polynomial x^2