# 6. More concepts in Metric Spaces

Slightly out of order from Sutherland.

(1) Infinite sequences in metric spaces
(2) Closed regions and subsets
(3) Associated closed and open subsets
(4) Reversible change of variables transformations

The definition of limit for a sequence $\{a_n\}$ in a metric space $(X,d)$ is the same as for $\mathbb{R}$.

$\lim_{n\to\infty} a_n = b \iff$ for each $\varepsilon > 0$ there is some $N \in \mathbb{N}^+$ such that $k \geq N \Rightarrow d(a_k, b) < \varepsilon$.

Just like proofs in $\mathbb{R}$.

**Prop. 6.26** There is at most one limit value.

**Ex. 6.25** $f: (X, d^X) \to (Y, d^Y)$ is continuous $\iff$ for all sequences, $\lim_{n\to\infty} a_n = a \Rightarrow \lim_{n\to\infty} f(a_n) = f(a)$.

the material between the green lines will not be covered on the exams.

Note. We shall not need Cauchy sequences in this course. These are sequence with the following property:

For each $\varepsilon > 0$ there is some $M$ such that $n, m \geq M \Rightarrow d(a_n, a_m) < \varepsilon$.

Ex. 6.24 A convergent sequence is Cauchy

The converse is false. Take $X = \mathbb{R} - \{0\}$ with the usual metric, $a_n = \frac{1}{n}$. Then $\lim_{n\to\infty} a_n$ does not exist in X. However,

if $(X, d)$ is a metric space, then X is isometric to a subset of a metric space $(X^*, d^*)$ in which every Cauchy sequence converges (see the Prop. on pp. 50–51 of grad-level-classnotes.pdf).

(2) CLOSED SUBSETS.

Standard domains for double integration have the form $a \le x \le b$, $g(x) \le y \le f(x)$ where $g + f$ are continuous. — These are the main examples.

Definition $(X, d)$ metric space, then $A \subseteq X$ is closed $\iff X - A$ is open.

Examples in $\mathbb{R}$

$[a, b]$, for $\mathbb{R} - [a, b] = (-\infty, a) \cup (b, \infty)$

$[a, \infty)$

$(-\infty, b]$

$A = [0,1] \cup [2,3]$, for $\mathbb{R} - A = (-\infty, 0) \cup (1,2) \cup (3, \infty)$

(Recall that open intervals are open in $\mathbb{R}$)

Note that a subset can be neither open nor closed. For example, $A = [0,1) \subseteq \mathbb{R}$.

Then $X - A = (-\infty, 0) \cup [1, \infty)$

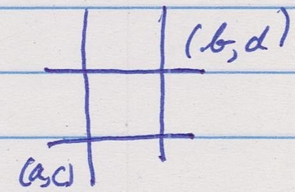
No set $N_r(t)$ is contained in $A$ if $t = 0$ or $X - A$ if $t = 1$. Therefore neither A nor X − A is an open subset.

More examples.

6.2.(a).(iv) $X = \mathbb{R}$, $A = \{\frac{1}{n} \mid n \in \mathbb{N}^+\} \cup \{0\}$,

for $X - A = (-\infty, 0) \cup (1, \infty) \cup \left( \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right) \right)$

(c) $X = \mathbb{R}^2$, $A = [a,b] \times [c,d]$, for




$$
\begin{aligned}
X - A = \ & (-\infty, a) \times \mathbb{R} \ \cup \\
& (b, \infty) \times \mathbb{R} \ \cup \\
& \mathbb{R} \times (-\infty, c) \ \cup \\
& \mathbb{R} \times (d, \infty)
\end{aligned}
$$

and $U$ open in $X$, $V$ open in $Y$ $\Rightarrow$

$U \times V = \pi_X^{-1}[U] \cap \pi_Y^{-1}[V]$ is open in the $d_p$ metric if $p = 1, 2, \infty$.

6.2.(d) If X has the discrete metric, then every subset is closed (since every subset is open).

<u>Variation on 6.2.(e)</u> If $f: X \to \mathbb{R}$ is continuous, then $f^{-1}[\{0\}]$ is closed, for $X - f^{-1}[\{0\}] = f^{-1}[\mathbb{R} - \{0\}]$ and $\mathbb{R} - \{0\} = (-\infty, 0) \cup (0, \infty)$ is open.

Similarly for $f^{-1}[[0,\infty)]$, $f^{-1}[(-\infty, 0]]$
[Complements are $(-\infty, 0)$ & $(0, \infty)$]

More generally,

<u>Prop. 6.6</u> $f: X \to Y$ is continuous $\Leftrightarrow$ for all closed $B \subseteq Y$, $f^{-1}[B]$ is closed in $X$.

<u>Idea of proof</u> Use $f^{-1}[Y-B] = X - f^{-1}[B]$. ∎

★ We want to show that if $\begin{matrix} f \\ g \end{matrix}: [a,b] \to \mathbb{R}$ are continuous, then $\{(x,y) \mid a \le x \le b,\ g(x) \le y \le f(x)\}$ is closed in $\mathbb{R}^2$.

<u>Preliminary steps</u> Additional exercise 5.7 says $f$ extends to all of $\mathbb{R}$ continuously.

In $\mathbb{R}^2$ we have
$$y \ge g(x) \Leftrightarrow y - g(x) \ge 0$$
$$y \le f(x) \Leftrightarrow f(x) - y \ge 0$$

so $\left.\begin{matrix} \{(x,y) \mid y - g(x) \ge 0\} \\ \{(x,y) \mid f(x) - y \ge 0\} \end{matrix}\right\}$ are closed since $\left.\begin{matrix} y - g(x) \\ f(x) - y \end{matrix}\right\}$ cont.

Likewise, $\{(x,y) \mid x \leq b\}$ and $\{(x,y) \mid a \leq x\}$ are closed.

Prop. 6. (a) $\phi$, $X$ are closed in $X$
(b) An arbitrary intersection of closed sets is closed.
(c) A finite (or 2 fold) union of closed sets is closed.

Proof. (a) $X - X = \phi$, $X - \phi = X$ and $\phi$, $X$ are open in $X$.
(b) $\{F_\alpha\}$ closed $\Rightarrow X - \cap F_\alpha = \bigcup_\alpha X - F_\alpha =$ union of open sets.
(c) $F_1 + F_2$ closed $\Rightarrow X - (F_1 \cup F_2) = (X - F_1) \cap (X - F_2) =$ intersection of two open sets. ∎

Hence $\{(x,y) \mid a \leq x \leq b, g(x) \leq y \leq f(x)\}$ is the intersection of $\{\ldots \mid a \leq x\}$, $\{\ldots \mid x \leq b\}$, $\{g(x) \leq y\}$ and $\{\ldots \mid y \leq f(x)\}$ and each of these subsets is closed. ∎

The origin of the named "closed" comes from the fact that $A \subseteq X$ is closed $\Leftrightarrow$ $A$ is "closed under taking limits of convergent sequences in $X$."