Sutherland, final sentence in Exercise 6.26

$A \subseteq X$ is closed $\Longleftrightarrow$ for each sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = b \in X$, the limit is in A.

Proof.

($\Longrightarrow$) Suppose $\lim_{n\to\infty} a_n = b \in X - A$; we are assuming that A is a closed subset. Choose $\varepsilon > 0$ so that $N_\varepsilon(b) \subseteq X - A$. By the definition of limit we have $a_n \in N_\varepsilon(b) \subseteq X - A$ for $n \geq N$. But this contradicts $a_n \in A$ for all n. Hence $\lim_{n\to\infty} a_n \in A$. ■

($\Longleftarrow$) Suppose A is not closed, so $X - A$ is not open. Then there is some $b \in X - A$ such that no set of the form $N_\varepsilon(b)$ is contained in $X - A$.

In other words, for each $\varepsilon > 0$ there is a point x_ε in $A \cap N_\varepsilon(b)$. Construct a sequence recursively: Pick $x_1 \in A \cap N_1(b)$. Given $\in x_n \in A$ s.t. $d(x_n, b) < \frac{1}{n}$, choose $x_{n+1} \in N_\varepsilon(b)$ where $\varepsilon = \min\{\frac{1}{2}d(x_n, b), \frac{1}{n+1}\}$.

Then $d(x_n, b) < \frac{1}{n}$ for all n, so $x_n \in A$ and $\lim_{n\to\infty} x_n = b \notin A$. ∎

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(3) <u>Note</u> If $A \subseteq \mathbb{R}^2$ is the closed region $a \le x \le b$, $g(x) \le y \le f(x)$, AND $g(x) < f(x)$ for $x \in (a,b)$, then $\{(x,y) \mid a < x < b,\ g(x) < y < f(x)\}$ is an <u>open</u> subset of $\mathbb{R}^2$.

<u>Proof.</u>

$$\left.\begin{array}{l} \{(x,y) \mid y > g(x)\} = \{\ldots \mid y - g(x) > 0\} \\ \{(x,y) \mid f(x) > y\} = \{\ldots \mid f(x) - y > 0\} \\ \{(x,y) \mid a < x < b\} \end{array}\right] \text{are all open sets}$$

(1st. two are inverse images of $(0,\infty)$ for cont. fcns. on $\mathbb{R}^2$), and the intersection of finitely many open sets is open.

<u>Suggests</u> In many cases, given a subset in $\mathbb{R}^2$ we have a naturally associated $\left\{\begin{array}{c}\text{open}\\ \text{closed}\end{array}\right\}$ subset.

$\boxed{\text{This is what we shall analyze next.}}$

<u>RELATED CONCEPT</u> <u>boundary</u> or <u>frontier</u> between $A \subseteq X$ and $X - A \subseteq X$.

$A \subseteq X$

<u>Closure</u> of $A = \bar{A}$, smallest closed set cont. $A =$

$$\bigcap_{\substack{F \text{ closed} \\ F \supseteq A}} F \quad \left(\begin{array}{c}\text{so there is a} \\ \text{unique smallest} \\ \text{set}\end{array}\right)$$

<u>Interior</u> of $A = \mathring{A} = \operatorname{Int} A$, largest open subset

contained in $A = \bigcup_{\substack{V \text{ open} \\ V \subseteq A}} V$ $\left(\begin{array}{c}\Rightarrow \text{a unique} \\ \text{largest set}\end{array}\right)$.

With hindsight, it's easier to take an indirect approach and start by looking at another concept.

<u>Definition 6.15</u> (X, d) metric space, $A \subseteq X$. Then $p \in X$ is a <u>limit point</u> of A if for each $\varepsilon > 0$, $(N_\varepsilon(p) - \{p\}) \cap A \neq \phi$. Notice that a point in A is not necessarily a limit point! $X = \mathbb{R}$, $A = \{0\}$ no limit points.

limit point = accumulation point

<u>Example</u> Limit points of (a, b) are $[a, b]$.

Explanation and verification.

First notice that in the definition, we can replace ε with all sufficiently small ε because $\varepsilon < \varepsilon' \Rightarrow N_\varepsilon \subseteq N_{\varepsilon'}$.

If $x \in (a,b)$ and $(x-\delta, x+\delta) \subseteq (a,b)$, then $(x-\delta, x+\delta)$ contains many points of (a,b) besides x itself.

If $x = a$ or b, and $\delta < b-a$, then $N_\delta(x)$ also contains infinitely many points of (a,b).

If $x < a$ or $x > b$, then $x \in (-\infty, a) \cup (b, \infty)$, which is open. Hence there is some $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq (-\infty, a) \cup (b, \infty)$ and hence $N_\varepsilon(x) \cap [a,b] = \phi$, so x is not a limit point.

$L(A)$ = set of limit points (= <u>derived set</u>).

<u>Proposition (Counterpart to Sutherland, 6.14)</u>

$A \subseteq X$ metric $\Longrightarrow$

(i) $L(A)$ is closed in X.

(ii) A is closed in $X \Longleftrightarrow L(A) \subseteq A$.

(iii) $A \cup L(A)$ is closed in X.

<u>Lemma</u> $x \in L(A) \iff$ there is some sequence $\{a_n\}$ in A such that $a_n \neq x$ all n and $x = \lim_{n \to \infty} a_n$.

<u>Proof of Lemma</u> ($\Rightarrow$) Take a_1 in $(N_1(x) - \{x\}) \cap A$, which is nonempty. Suppose we have a_k for $k \leq n$ with $d(x, a_k) < \frac{1}{k}$. Let $\varepsilon = \min\{d(x, a_k), \frac{1}{k+1}\}$. Pick $a_{k+1} \in (N_\varepsilon(x) - \{x\}) \cap A$. Then $a_n \to a$. ∎ (as in proof on p. 6.6 of the notes)

($\Leftarrow$) Given $\varepsilon > 0$, choose N such that $n \geq N \Rightarrow d(x, a_n) < \varepsilon$. Then $a_n \in (N_\varepsilon(x) - \{x\}) \cap A$. ∎

<u>Cor.</u> If $x \in L(A)$, then every $N_\varepsilon(x) - \{x\}$ contains <u>infinitely many</u> points of A.
(Look at the a_n in the sequence!!).

<u>Note also</u> that $B \subseteq A \Rightarrow L(B) \subseteq L(A)$.

<u>PROOF OF PROPOSITION</u>. First prove (ii), then (i), then (iii).