Remark on limit sets

The following proof, which was given in the lectures, provides a simpler argument for proving the a property of limit sets on page 6.09 of `math145Anotes06b.pdf`:

Let X be a metric space, let $A \subset X$ and if B is an arbitrary subset of X let $\mathbf{L}(B)$ denote the set of limit points of B (in X). Then we have $\mathbf{L}(\mathbf{L}(A)) \subset \mathbf{L}(A)$.

Proof. Suppose that $p \in \mathbf{L}(\mathbf{L}(A))$, and let $\varepsilon > 0$. Then we know that $N_\varepsilon(p)$ contains a point $q \neq p$ such that $q \in \mathbf{L}(A)$. Now let $\delta > 0$ satisfy $\varepsilon - d(p, q)$; by construction we have $p \notin N_\delta(q)$ and $N_\delta(q) \subset N_\varepsilon(p)$. Since $q \in \mathbf{L}(A)$ we also know that $N_\delta(q)$ contains some point $y \in A$. By the preceding reasoning we know that $y \neq p$ (the first is in the δ-neighborhood while the second is not), and furthermore we have

$$y \quad \in \quad N_\delta(q) \quad \subset \quad N_\varepsilon(p) \ .$$

The statements in the previous sentence now imply that $p \in \mathbf{L}(A)$, and hence that $\mathbf{L}(\mathbf{L}(A)) \subset \mathbf{L}(A)$.∎

As usual, the reader is encouraged to draw a rough sketch in order to "see" the ideas behind the argument.