(ii) ($\Longrightarrow$) Suppose A is closed, so that $X-A$ is open. Want to show $L(A) \cap X-A = \phi$.

But $y \in X-A \Rightarrow$ some $N_\varepsilon(y) \subseteq X-A$

$\Rightarrow (N_\varepsilon(y) - \{y\}) \cap A = \phi \Rightarrow y \notin L(A)$. ∎

($\Longleftarrow$) Suppose $L(A) \subseteq A$. Want to show $X-A$ is open in X. But then $y \in X-A \Rightarrow$ $y \notin L(A) \Rightarrow$ for some $\varepsilon > 0$, $(N_\varepsilon(y) - \{y\}) \cap A$ $= \phi$, so that also $N_\varepsilon(y) \cap A = \phi$ or $N_\varepsilon(y) \subseteq X-A$. ∎

(i) By (ii), we need only show that $L(L(A)) \subseteq L(A)$. Suppose we have $b_n \to c$ where $b_n \neq c$ and $b_n \in L(A)$. Then we can find $a_n \neq b_n$ s.t. $a_n \in A$ and $d(a_n, b_n) < \begin{cases} \frac{1}{n} \\ d(b, b_n) \end{cases}$. Claim $a_n \neq b$

$a_n \to b$.

$a_n \neq b$ because $d(a_n, b_n) < d(b, b_n)$

To verify $a_n \to b$, let $\varepsilon > 0$.

Then $d(b_n, b) < \frac{\varepsilon}{2}$ if $n \geq M_1$

$d(a_n, b_n) < \frac{\varepsilon}{2}$ if $n \geq M_2$ $\Rightarrow$

$\varepsilon > 0$
$\Rightarrow$
$\frac{\varepsilon}{2} > 0$
"trick"

if $n \geq \max\{M_1, M_2\}$ then $d(a_n, b) \leq$
$d(a_n, b_n) + d(b_n, b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. ∎

(iii) Show the complement is open.

Let $y \notin A \cup L(A)$ but $y \in X$. Since $y \notin L(A)$, for some $\varepsilon > 0$ $(N_\varepsilon(y) - \{y\}) \cap A = \phi$, and in fact $N_\varepsilon(y) \cap A = \phi$ because $y \notin A$.

Proceed to page 12A

CLAIM: $N_\varepsilon(x) \cap L(A) = \phi$ too.

If so, then $N_\varepsilon(x) \cap (A \cup L(A)) = \phi$, which implies that $X - (A \cup L(A))$ is open.

But suppose $y \in N_\varepsilon(x) \cap L(A)$ and take $r > 0$ so that $N_r(y) \subseteq N_\varepsilon(x)$. Then $(N_r(y) - \{y\}) \cap A \neq \phi$, so that $N_\varepsilon(x) \cap A \neq \phi$, contradiction. The source of the contradiction was the assumption that $N_\varepsilon(x) \cap L(A) \neq \phi$, so $N_\varepsilon(x) \cap A = \phi$ as claimed. ∎

Corollary $\bar{A} = A \cup L(A)$.

Why F for a closed set? The French term is fermé.

Proof. $\bar{A} \subseteq A \cup L(A)$ since the right side is closed in X and it contains A. Conversely, if $A \subseteq F$ closed, then $L(A) \subseteq L(F) \subseteq F$, so $A \cup L(A) \subseteq F$. Hence $A \cup L(A) \subseteq \bigcap_{\substack{F \supseteq A \\ \text{closed}}} F = \bar{A}$. ∎

Strictly speaking, $A \cup L(A) = Cl(A; X)$, the closure of A in X.

More on closures, etc.

A is dense in X if $\overline{A} = X$

X is *dense-in-itself* if $X = L(X)$.

Sutherland, Prop. 6.11 (partial), 6.13, 6.14:

(i) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$

(ii) $\overline{\overline{A}} = \overline{A}$

(iii) $\overline{\bigcap_\alpha A_\alpha} \subseteq \bigcap_\alpha \overline{A_\alpha}$

(iv) $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$.

Note: Containment in (iii) may be proper. Let $X = \mathbb{R}$, $A_1 = (-\infty, 0)$, $A_2 = (0, \infty)$ so

$A_1 \cap A_2 = \phi \Rightarrow A_1 \cap A_2 = \overline{A_1 \cap A_2}$, but

$0 \in \overline{A_1} \cap \overline{A_2}$.

Proofs. (i) $A \subseteq B \subseteq \overline{B}$, so $\overline{B}$ is a closed subset containing A, so $\overline{A} \subseteq \overline{B}$. $\blacksquare$

(ii) $\overline{\overline{A}}$ is a closed set containing $\overline{A}$, so
$\overline{A} \subseteq \overline{\overline{A}}$, and $\overline{\overline{A}} = \overline{A} \cup L(\overline{A}) \subseteq \overline{A} \cup \overline{A} = \overline{A}$. ■

(iii) RHS is a closed subset containing the intersection, so it also contains the latter's closure. ■

(iv) By (i), $\overline{A_i} \subseteq \overline{A_1 \cup A_2}$, so
$\overline{A_1} \cup \overline{A_2} \subseteq \overline{A_1 \cup A_2}$. But $\overline{A_1} \cup \overline{A_2}$ is closed and contains $A_1 \cup A_2$, so also
$$\overline{A_1 \cup A_2} \subseteq \overline{A_1} \cup \overline{A_2}. \blacksquare$$

<u>Sutherland, Prop. 6.12</u> $f : (X, d^X) \to (Y, d^Y)$ is continuous $\iff$ for all $A \subseteq X$ we have
$f[\overline{A}] \subseteq \overline{f[A]}$. (Proved in Exercise 6.13).

<u>INTERIOR OF $A \subseteq X$</u> $\mathring{A}$, A°, or
$\text{Int}(A, X)$, or sometimes $\text{Int}(A)$ – [if no ambiguity about X]

Formally, it is all $a \in A$ such that some $N_\varepsilon(a) \subseteq A$.