<u>EXAMPLES</u> 1. The interior of $[a,b]$ is (a,b).

If $x \in (a,b)$, then some $(x-\delta, x+\delta) \subseteq (a,b) \subseteq [a,b]$, so $(a,b) \subseteq$ Interior.

On the other hand, for each $\varepsilon > 0$ the sets $N_\varepsilon(a)$, $N_\varepsilon(b)$ are <u>not</u> contained in $[a,b]$, so $a, b \notin \mathrm{Int}([a,b])$.

2. The interior of $\{0\}$ in $\mathbb{R}$ is empty, ~~for~~ $N_\varepsilon(0)$ is never cont. in $\{0\}$.

3. The interior of $\mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ is also empty for the same reason (here, $N_\varepsilon((t,0))$ contains $(t, \varepsilon/2)$).

<u>CLAIM</u> $\mathrm{Int}(A, X)$ is the unique largest open subset $U \subseteq X$ such that $U \subseteq A$.

<u>Verification</u> (i) U open $+ U \subseteq A \Rightarrow U \subseteq \mathrm{Int} A$, for if $a \in U$ open & $U \subseteq A$, then for some $\varepsilon > 0$ we have $N_{\varepsilon(a)}(a) \subseteq U \subseteq A$, and hence $a \in \mathrm{Int} A$. ∎

(ii) We shall prove $\mathrm{Int}(A, X)$ is open in X.

Let $a \in \mathrm{Int}(A, X)$, $N_{\varepsilon(a)}(a) \subseteq A$; we shall show $N_\varepsilon(a) \subseteq \mathrm{Int} A$. But $y \in N_{\varepsilon(a)}(a)$ and $\delta = d(a, y) \Rightarrow N_{\varepsilon(a)-\delta}(y) \subseteq N_{\varepsilon(a)}(a) \subseteq A$, so $y \in \mathrm{Int} A$. Finally, we use this to see

$$\mathrm{Int} A = \bigcup_{a \in \mathrm{Int} A} \{a\} \subseteq \bigcup_{a \in \mathrm{Int} A} N_{\varepsilon(a)}(a) \subseteq \mathrm{Int} A$$

so that $\mathrm{Int}\ A$ is the open set $\bigcup_{a \in \mathrm{Int} A} N_{\varepsilon(a)}(a)$. ∎

<u>More on interiors (Sutherland, 6.21)</u>

(i) $A \subseteq B \Rightarrow \mathrm{Int}(A) \subseteq \mathrm{Int}(B)$.

(ii) $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$.

Proofs

(i) $\mathrm{Int}(A) \subseteq A \subseteq B$, so $\mathrm{Int}(A)$ is an open set contained in B. Since $\mathrm{Int}(B)$ is the largest such open set, $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$. ■

(ii) $\mathrm{Int}(A) \subseteq A$ & (i) imply

$\mathrm{Int}(\mathrm{Int}(A)) \subseteq \mathrm{Int}(A)$. To finish, note that if U is open, then $\mathrm{Int}(U) = U$ follows from the definitions. ■

Boundary or frontier points.

Warning In topology, ∂A often has another meaning, and similarly for boundary.

Def. 6.22 $A \subseteq X$ metric. The frontier or boundary of A in X, written $\mathrm{Bdy}(A;X)$ or $\mathrm{Fr}(A;X)$ is $\overline{A} - \mathrm{Int}(A)$.

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Example. The frontier of (a,b) or $[a,b]$ in $\mathbb{R}$ is equal to $\{a, b\}$.

Examples

1. The frontier points of $[a,b]$ and (a,b) are $\{a,b\}$.

2. The frontier points of $\mathbb{Q}$ are all of $\mathbb{R}$, for $\overline{\mathbb{Q}} = \mathbb{R}$ (since $L(\mathbb{Q}) = \mathbb{R}$!) and no set of the form $N_\varepsilon(q)$ is contained in $\mathbb{Q}$ (between q and $q+\varepsilon$ there is some irrational number).

Sutherland, Prop. 6.24 Let $x \in A \subseteq X$ metric. Then $x \in \mathrm{Bdy}(A,X) \iff$ for all $\varepsilon > 0$ both $A \cap N_\varepsilon(x)$ and $(X-A) \cap N_\varepsilon(x)$ are nonempty.

Proof. $(\Rightarrow)$ $x \in \mathrm{Bdy}(A,X) \Rightarrow x \in \overline{A} - \mathrm{Int} A$. Since $x \in \overline{A}$, for each $\varepsilon > 0$ the set $N_\varepsilon(x) \cap A$ is nonempty (two cases: $x \in A$, $x \in L(A)$). Since $x \notin \mathrm{Int} A$, there is no $\varepsilon > 0$ s.t. $N_\varepsilon(x) \subseteq A$. In other words, for each $\varepsilon > 0$ there is at least one point in $N_\varepsilon(x) \cap (X-A)$. ∎

($\Leftarrow$) If $N_\varepsilon(x) \cap A$ is non-empty for all ε, then either $x \in A$ or $x \in L(A)$, so $x \in \bar{A}$.
If $N_\varepsilon(x) \cap (X - A) \neq \phi$ for all ε, then $x \notin \text{Int} A$. ∎

<u>EQUIVALENT METRICS</u> Postponed until next chapter with one exception.

<u>Def.</u> (X, d^X) + (Y, d^Y) metric spaces.
$f : (X, d^X) \to (Y, d^Y)$ is a <u>homeomorphism</u>

(home - ee - oh - MORE - fizm)

if f is 1-1 onto and both f and its inverse are continuous.

<u>THIS IS A FUNDAMENTAL CONCEPT IN TOPOLOGY!</u>

Special cases have arisen in earlier courses.

<u>Single variable calculus</u> Change of variables.

$u(x)$ continuous derivative, strictly increasing on $[a, b] \Longrightarrow u$ has a continuous derivative and a strictly increasing inverse.

One use is to rewrite integrals into more computable forms: