8. Continuity & Bases

Def. $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ topological spaces. Then $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous if for each V open in Y the set $f^{-1}[V]$ is open in X.

(Compatible with the def. for metric spaces) [Sutherland, Prop. 8.3]

Sutherland, Prop. 8.4 A composite of continuous functions is continuous.

Proof. $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$ continuous.

If W is open in Z, then $g^{-1}[W]$ is open in Y since g is continuous. Likewise, the set $f^{-1}[g^{-1}[W]]$ is open in X. Since the latter is $(g \circ f)^{-1}[W]$, $g \circ f$ is continuous. ∎

Sutherland, Prop. 8.6 (a) The identity map $1_X : (X, \mathcal{T}_X) \to (X, \mathcal{T}_X)$ is continuous.

(c) Every $f : (X, \text{discrete}) \to (Y, \mathcal{T}_Y)$ is continuous.

(b) A constant map is continuous.

(d) Every $f : (X, \mathcal{T}_X) \to (Y, \text{indiscrete})$ is continuous.

(e) Every $f : (\emptyset, \{\emptyset\}) \to (X, \mathcal{T}_X)$ is continuous.
(empty set is a topological space!)

<u>Bases = Topologies generated by families of subsets</u>

<u>Observation</u>: An intersection of topologies on X is a topology on X.

If $\mathcal{O}$ is a family of subsets of X, let $\mathcal{T}(\mathcal{O})$ = intersection of all topologies $\mathcal{T}$ on X such that $\mathcal{O} \subseteq \mathcal{T}$. (Nonempty – e.g. $\mathcal{T}$ = discrete).

<u>Proposition</u> $\mathcal{T}(\mathcal{O})$ consists of $\emptyset$, X and arbitrary unions of finite intersections of sets in $\mathcal{O}$.

<u>Proof.</u> All such sets lie in $\mathcal{T}(\mathcal{O})$, so we need to show the conditions define a topology.

The only non-immediate part is to show the family described in the Proposition is closed under intersections (note that "a union of unions is a union" GERTRUDE STEIN PROPERTY)

But $\left(\bigcup_\alpha A_{\alpha_1} \cap \dots \cap A_{\alpha_{k(\alpha)}}\right) \cap \left(\bigcup_\beta A_{\beta_1} \cap \dots \cap A_{\beta_{\ell(\alpha)}}\right)$

$= \bigcup_{\alpha,\beta} A_{\alpha_1} \cap \dots \cap A_{\alpha_k} \cap A_{\beta_1} \cap \dots \cap A_{\beta_\ell}$. ∎

$\boxed{(X, \mathcal{T}) \text{ top space}}$

Def. $\mathcal{B} \subseteq \mathcal{T}$ is a base for $\mathcal{T}$ if every open set is a union $\cup U_\alpha$ with $U_\alpha \in \mathcal{B}$.

Examples (X,d) metric and
$\mathcal{B} = N_\varepsilon(x)$ where $x \in X$ and $\begin{cases} \varepsilon > 0 \\ \varepsilon = 1/n \text{ some } n. \end{cases}$

$(X, \text{discrete})$ $\mathcal{B} = \text{all } \{x\}$.

One useful property of bases:

Sutherland, Prop. 8.12 $\mathcal{B}$ = base for $\mathcal{T}_Y$. Then $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous $\Longleftrightarrow$ $f^{-1}[V]$ is open for all V in $\mathcal{B}$.

PROOF. ($\Rightarrow$) Trivial since we know $V \in \mathcal{T}_Y \Rightarrow f^{-1}[V] \in \mathcal{T}_X$ and $\mathcal{B} \subseteq \mathcal{T}_Y$.

($\Leftarrow$) $V \in \mathcal{T}_Y \Rightarrow V = \bigcup_\alpha U_\alpha$ for $U_\alpha \in \mathcal{B}$.

Hence $f^{-1}[V] = f^{-1}[\cup_\alpha U_\alpha] = \cup_\alpha f^{-1}[U_\alpha]$, which is a union of open sets and hence is open. $\boxtimes$

Note If $\mathcal{T} = \mathcal{T}(\mathcal{O})$, say $\mathcal{O}$ is a sub-base for $\mathcal{T}$.

Homeomorphisms, Part A

$$f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y) \quad \text{s.t.}$$

f is 1-1 onto, f and f^{-1} both continuous.

(See Additional Exercise 8.2 for more)

Examples (a, b) homeo (c, d)
$[a, b]$ homeo $[c, d]$
$(-1, 1)$ homeo $\mathbb{R}$

First two $f(x) = c + \dfrac{(d-c)}{(b-a)}(x-a)$ PRE CALC!

Last one $f(x) = \dfrac{x}{1-|x|}$ OR $\tan \frac{\pi}{2} x$

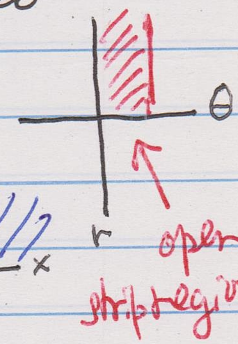
∞ DIFFERENTIABLE

Homeomorphisms, Part B Geometrically, homeomorphic subsets of $\mathbb{R}^n$ resemble each other in some non-rigid sense.

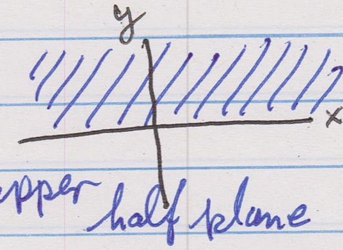
For example, think about polar coords

$$x = r\cos\theta$$
$$y = r\sin\theta$$

These define a homeo from $\{(r,\theta)\in\mathbb{R}^2 \mid r>0,\ 0<\theta<\pi\}$



to $\{(x,y)\in\mathbb{R}^2 \mid y>0\}$



Inverse map:

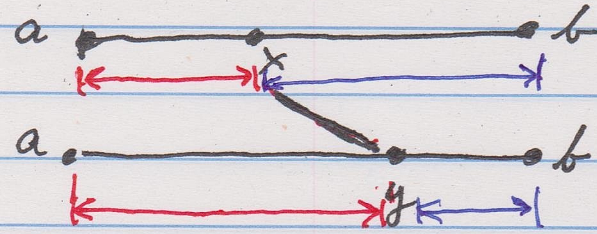
$$r=\sqrt{x^2+y^2},\quad \theta=\arccos\frac{x}{\sqrt{x^2+y^2}}.$$

Go to intro2topA-08.pdf and intro2topA-08a.pdf for more.

The geometric behavior of homeomorphisms is behind the frequently repeated description of topology as a "rubber sheet geometry."

Here is another simple example to illustrate how homeomorphisms can bend and stretch a space.

Let $X = [a, b]$ with $a < x < y < b$.
Then there is a homeomorphism $f: X \to X$ such that $f(a) = a$, $f(x) = y$, $f(b) = b$.



f stretches $[a, x]$ to $[a, y]$ linearly, and f shrinks $[x, b]$ to $[y, b]$ linearly.

Note that the restriction of f to (a, b) yields a homeomorphism h from (a, b) to itself such that $h(x) = y$. So given two points of (a, b) or there is a homeomorphism h from (a, b) to itself which sends one point to the other.

See the first page of intro2topA-12c.pdf for detailed proofs of the assertions in the discussion above.