9. Some concepts in Topological Spaces

Main purpose Modify various concepts and results from Chapter 6 so they are meaningful for topological spaces.

Two major differences $(X, \mathcal{T})$ top space

(1) If $\mathcal{T}$ does not come from a metric, one point subsets are not necessarily closed. (Sierpiński topology, etc.)

(2) In a metric space, if $x \in X$ then there are open subsets $U_1 \supseteq U_2 \supseteq \ldots$ such that $\{x\} = \bigcap_{n=1}^{\infty} U_n$: Take $U_n = N_{1/n}(x)$.

They arise naturally in some contexts

There are important examples of topological spaces which do not have this property.

<u>Example for (2)</u> [maybe not the most natural, but simple to work with].

X = uncountable set, say $\mathbb{R}$ or $\mathbb{C}$, with $\mathcal{T}$ = cofinite topology. — If we have open sets $U_1 \supseteq U_2 \ldots$ with $x \in U_i$ for all i, then $X - \bigcap_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} X - U_i = \bigcup_{i=1}^{\infty} F_i$ where F_i is finite, so $X - \bigcap_{i=1}^{\infty} U_i$ is countably infinite and hence $\bigcap_{i=1}^{\infty} U_i \neq \{x\}$.

The preceding two facts have the following significance:

(1) If $f: X \to Y$ is continuous, then the "level sets" $f^{-1}[\{y\}]$ need not be closed.

MAJOR POINT

(2) We cannot use ordinary convergent ∞ sequences when trying to prove results about topological spaces which might not come from metric spaces.

<u>NOTE</u>: There is a more general notion of (Moore-Smith) convergence which works in arbitrary topological spaces, but it is not as easy to work with in most cases, and it is much less widely used than ordinary convergence in metric spaces.

this is a little subjective

For example, M-S convergence is not covered in most graduate level topology courses, and has not been for several decades.

<u>Sutherland, Prop. 9.5</u> $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ is continuous $\Leftrightarrow$ for each closed subset $F\subseteq Y$, the inverse image $f^{-1}[F]$ is closed in X.

<u>Proof.</u> Suppose f is continuous. Then F closed in $Y \Rightarrow Y-F$ open in $Y \Rightarrow$ $f^{-1}[Y-F]=X-f^{-1}[F]$ open in $X \Rightarrow$ $f^{-1}[F]$ closed in X. Conversely, if the condition in the proposition holds, then U open in $Y \Rightarrow Y-U$ closed in $Y \Rightarrow$ $f^{-1}[Y-U]=X-f^{-1}[U]$ closed in $X \Rightarrow$ $f^{-1}[U]$ open in X. ∎

Here is the main thing we need:

<u>Def.</u> X top space, $A\subseteq X$. Then $p\in X$ is a limit point of $A \Leftrightarrow$ for each open set U containing p we have

$$(U-\{p\})\cap A\neq\emptyset.$$

CLAIM For metric spaces, this is equivalent to the property in the previous definition.

$$\boxed{\text{For each } \varepsilon > 0,\ (N_\varepsilon(p) - \{p\}) \cap A \neq \phi}$$

NEW ⇒ OLD $N_\varepsilon(p)$ is an open set containing p

OLD ⇒ NEW Given U, choose $\delta > 0$ so that $N_\delta(p) \subseteq U$. Then $(N_\delta(p) - \{p\}) \cap A \neq \phi$ implies $(U - \{p\}) \cap A \neq \phi$.

Modified counterpart to Sutherland, 6.14

$A \subseteq X$ top space $\Rightarrow$

(ii) A is closed in $X \Leftrightarrow L(A) \subseteq A$

(iii) $A \cup L(A)$ is closed in X

(i)* If $\{p\}$ is closed for all $p \in X$, then $L(A)$ is closed in X.

The proofs of (ii) & (iii) from Chapter 6 go through if we replace the metric neighborhoods $N_\varepsilon(p)$ with arbitrary open sets containing p.

However, we need the following new proof for (i)*:

It suffices to show $L(L(A)) \subseteq L(A)$.

Suppose $p \in L(L(A))$. Let U be open with $p \in U$. Then there is some point $q \in (U - \{p\}) \cap L(A)$. By hypothesis $U - \{p\}$ is open, so $q \in L(A) \Rightarrow$ there is some point $q' \in (U - \{p,q\}) \cap A \subseteq (U - \{p\}) \cap A$. ∎

The whole discussion of closures now goes through for topological spaces (taking earlier points (1) and (2) into account).

<u>Interiors</u> Redefine $\mathrm{Int}(A, X)$ as all $a \in A$ such that there is some open set U with $a \in U \subseteq A$.

As before, this reduces to the earlier definition for metric spaces. The main adjustment needed involves the proof that $\mathrm{Int}(A,X)$ = largest open set $\subseteq X$.

page 6.15

<u>Changes needed</u> (i) U open and $U \subseteq A \Rightarrow U \subseteq \mathrm{Int}\,A$. — No need to talk about N_ε's.

(ii) Prove $\mathrm{Int}(A, X)$ is open. If $a \in \mathrm{Int}(A, X)$ and $a \in U_a$ open with $U_a \subseteq A$, then $b \in U_a \Rightarrow b \in \mathrm{Int}\,A$, so $U_a \subseteq \mathrm{Int}(A)$. Hence

$$\mathrm{Int}\,A = \bigcup_{a \in A} \{a\} \subseteq \bigcup_{a \in A} U_a \subseteq \mathrm{Int}\,A \text{ as before. } \blacksquare$$

so A is a union of open sets.

<u>Modified Sutherland, Prop. 6.24</u> [corrected] $x \in X$, $A \subseteq X$. Then $x \in \mathrm{Bdy}(A, X) \Leftrightarrow$ for all open sets U with $x \in U$, both $A \cap U$ and $(X - A) \cap U$ are nonempty. $\blacksquare$

[Again, replace $N_\varepsilon(x)$'s with U's.]

<u>Sutherland, Def. 9.22</u> $x \in N \subseteq X$. Say N is a neighborhood of x if there is an open set U in X <u>such that</u> $x \in U \subseteq N$.

$\Rightarrow$ Not necessarily open! For example, $[a-h, a+h]$ is a <u>closed</u> neighborhood of a in $\mathbb{R}$ for each $h > 0$. $\leftarrow$ $\boxed{\text{N can be neither open nor closed}}$