10. SUBSPACES AND PRODUCT SPACES

More generalizations from metric to topological spaces.

Def. $(X, \mathcal{T})$ topological space, $A \subseteq X$. Then $\mathcal{T}|A$ ($\mathcal{T}$ restricted to A) is all sets $V \cap A$, where V is open in X.

We should check this is a topology, and if $\mathcal{T}$ comes from a metric d, then $\mathcal{T}|A$ comes from $d|A \times A$. First part is Exe. 10.2, the second is Exc. 10.4.

Sutherland, 10.4–10.6

(i) The inclusion $j = j_{A \subseteq X} : A \to X$ is continuous.

(ii) $f : X \to Y$ continuous $\Rightarrow f|A = f \circ j : A \to X$ is continuous.

restriction of f to A

(iii) If $f[X] \subseteq B$, then the function $g : X \to B$, $g(x) = f(x)$, is continuous.

Proofs. (i) U open in $X \Rightarrow j^{-1}[U] = U \cap A$.

(ii) $f|A$ is a composite of cont. fcns.

(iii) Suppose V is open in B, and write $V = B \cap U$ where U is open in Y. Then $f^{-1}[U]$ is open. However, $f[X] \subseteq B \Rightarrow$ $f^{-1}[U] = f^{-1}[U \cap B] = g^{-1}[V]$, so the latter is open in X. ■

<u>Gertrude Stein Rules</u> (An $\boxed{*}$ of a $\boxed{*}$ is a $\boxed{*}$).

"WILD CARD"

B is $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ in A (with respect to subspace topology) and A is $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ in $X \Rightarrow B$ is too.

<u>Proof.</u> Write $B = C \cap A$ where C is $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ in X. Since a finite intersection of $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ sets is $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$, this means B is $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ in X. ■

<u>PRODUCT TOPOLOGIES</u> $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ top spaces. Then the product topology is generated by all sets $U \times V$, where $\left\{\begin{matrix}U \in \mathcal{T}_X\\ V \in \mathcal{T}_Y\end{matrix}\right.$.

If $\mathcal{T}_X$ and $\mathcal{T}_Y$ come from metrics, this is the d_∞ product metric topology by definition. Hence all d_p metrics define the product topology on $X \times Y$. ($p = 1, 2, \infty$)

Note Not every set in the product top. is a finite union of rectangular open sets. See intro2topA-10.pdf for more.

Products and continuous functions

$\pi_X, \pi_Y : X \times Y \longrightarrow X, Y$ are continuous

$$\pi_X^{-1}[U] = U \times Y, \quad \pi_Y^{-1}[V] = X \times V,$$

so inverse images of open sets are open.

Sutherland, Prop 10.11 modified X, Y, W top spaces, $f : W \longrightarrow X$ and $g : W \longrightarrow Y$ cont. Define a set theoretic function $h : W \longrightarrow X \times Y$ by $h(w) = (f(w), g(w))$. Then h is continuous.

Proof. (1) Show inverse images of basic open subsets are open. But $h^{-1}[U \times V] = f^{-1}[U] \times g^{-1}[V]$. If f, g continuous, this is a product of open sets and hence is open.

(2) General case. We can write $D = \cup\, U_\alpha \times V_\alpha$ if D is open in $X \times Y$,* and $h^{-1}[D] = \bigcup_\alpha h^{-1}[U_\alpha \times V_\alpha]$. By (1) the RHS is a union of open subsets and hence the union is open in W. ∎

* see Sutherland Prop. 10.9

Sutherland, 10.14 Given (x_0, y_0) in $X \times Y$, the slice inclusions $i(x_0): Y \longrightarrow X \times Y$, $y \longrightarrow (x_0, y)$ and $j(y_0): X \longrightarrow X \times Y$, $x \longrightarrow (x, y_0)$ define homeomorphisms onto their images.

Proof. The maps are 1-1 onto, and they are continuous because $\{f, g\} = \{\text{identity, constant}\}$ in each case. Check directly that inverses are given by projection onto $\left\{ \begin{matrix} Y \\ X \end{matrix} \right\}$ for $\left\{ \begin{matrix} i \\ j \end{matrix} \right\}$. ∎

More precise version

Let $i' : Y \longrightarrow \{x_0\} \times Y$ be induced by
$\quad j' : X \longrightarrow X \times \{y_0\}$ i and j.

Then i' + j' are 1-1, onto, and continuous.

The continuous inverses are the mappings

$\pi_Y \mid \{x_0\} \times Y$
$\pi_X \mid X \times \{y_0\}$ respectively. ∎

<u>Sutherland, 10.19</u> $f: X \to Y$ cont.
Γ_f = graph of f = all $(x,y) \in X \times Y$ so that $y = f(x)$. Then Γ_f is homeomorphic to X.

<u>Proof.</u> Define $h: X \longrightarrow X \times Y$ s.t. $\pi_X h = id_X$, $\pi_Y h = f$, so h is continuous. It is 1-1 since $h(x) = h(x') \Rightarrow x = \pi_X h(x) = \pi_X h(x') = x'$. Let $h_0 : X \longrightarrow \Gamma_f$ be the induced 1-1 onto continuous map, and let $k_0 : \Gamma_f \to X$ be $\pi_X | \Gamma_f$. Then k_0 is a continuous inverse to h_0. ∎

<u>Comment</u> If X and Y come from metric spaces, then $\Gamma_f \subseteq X \times Y$ is closed. Generalizations to top spaces are discussed in Chapter 11.

<u>Read</u> <u>Sutherland 10.20</u>: A useful tool for showing subsets in $X \times Y$ are open in some cases.