~

Detailed arguments for some assertions in Chapter 10

Alternate characterization of the subspace topology

The proof of the following result was not finished in the notes, so here is a complete argument:

PROPOSITION C. *Let $B \subset A \subset X$ where X is a topological space. Then B is closed with respect to the subspace topology on A if and only if $B = A \cap F$ where F is a closed subset of X.*

Proof. B is closed in A if and only if $A - B$ is open in A, which by definition is true if and only if $A - B = A \cap V$ where V is open in X.

Regardless of whether V is open in X, if $A - B = X \cap V$ we have

$$B \;=\; A - (A - B) \;=\; A - (X \cap V) \;=\; (X - V) \cap A \; .$$

Combining these two observations, we see that B is closed in A if and only if $B = (X - V) \cap A$ for some open subset $V \subset X$, which is equivalent to saying that $B = F \cap A$ where F is a closed subset of X.■

Base for the product topology

We have defined the subspace topology on a product $X \times Y$ to be the topology generated by all subsets of the form $U \times V$, where U is open in X and V is open in Y. It turns out that this family of subsets is actually a **base** for the product topology and on page 4 of `math145Anotes10.pdf` we give a marginal reference to Sutherland for this fact. The proof is a direct consequence of two observations:

Observation 1. *The family $\mathcal{P}$ of all subsets of the form $U \times V$, where U is open in X and V is open in Y, is closed under finite intersections.*

Observation 2. *If $\mathcal{F}$ is a subbase for the topology $\mathcal{T}$ and $\mathcal{F}$ is closed under taking finite intersections, then $\mathcal{F}$ is a base for $\mathcal{T}$.*

If we combine these with the definition of product topology as the topology generated by $\mathcal{P}$, we see that the latter is in fact a base for $\mathcal{T}$.■

Verification of Observation 1. By induction it will suffice to verify that the family $\mathcal{P}$ is closed under twofold intersections (explain this in more detail!). The property for twofold intersections follows from the set-theoretic identity

$$(U \times V) \;\cap\; (U' \times V') \quad=\quad (U \cap U') \;\times\; (V \cap V')$$

the fact that the families of open sets for the topologies of both X and Y are closed under taking the intersections of subsets in the respective families.■

Verification of Observation 2. The topology $\mathcal{T}$ is given by arbitrary unions of finite intersections of subsets in $\mathcal{F}$, and we are asuming that the latter is closed under taking finite intersections, so it follows that every open set in $\mathcal{T}$ is a union of sets in $\mathcal{F}$. But this condition is the definition of a base for the topology $\mathcal{T}$.■