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More detailed arguments for assertions in Chapter 10

The subspace topology for metric spaces

If (X, d) is a metric space and A is a subspace of X, then one way to make A into a topological space is to take the restriction of the metric d to $A \times A$ and use the topology A inherits from this metric. Another way to make A into a topological space is to use the definition of the subspace topology: $U \subset A$ is open in this topology if and only if $U = V \cap A$, where V is open in X. Our purpose here is to verify that these topologies are the same.

To make the discussion more precise, if $X \in A$ and $\varepsilon > 0$, then $N_\epsilon(x; X)$ will denote its neighborhood of radius ε in X and $N_\epsilon(x; A)$ will denote the corresponding neighborhood in A. By definition we have $N_\epsilon(x; A) = A \cap N_\epsilon(x; X)$.

If a subset is open in the restricted metric topology on A, then it is open in the subspace topology on A. If W is open in the restricted metric topology, then we have

$$W = \bigcup_{a \in W} N_{\varepsilon(a)}(a; A)$$

where $\varepsilon(a)$ is chosen so that $N_{\varepsilon(a)}(a; A) \subset W$.

If we now let

$$U = \bigcup_{a \in A} N_{\varepsilon(a)}(a; X)$$

then by construction (and the distributive law for intersections over unions) we see that U is open and $A \cap U = W$, which means that W is open in the subspace topology.■

If a subset is open in the subspace topology on A, then it is open in the restricted metric topology on A. Suppose that $W = A \cap U$ where U is open in X. Much as before we have

$$U = \bigcup_{x \in U} N_{\varepsilon(x)}(x; X) .$$

Now let $V \subset U$ be the open subset in X for which one takes the union only over points in $W = A \cap U$. Then $V \subset U$ and hence $A \cap V \subset A \cap U = W$ on one hand, but on the other hand since $x \in N_{\varepsilon(x)}(x; X)$ for all x we also have $W \subset V$ by the construction for V. The latter implies that $W = A \cap W \subset A \cap V$, so that $W = A \cap V$ where the latter is open in X. Finally, we have

$$W = A \cap V = A \cap \left(\bigcup_{x \in W} N_{\varepsilon(x)}(x; X) \right) =$$

$$\bigcup_{x \in W} A \cap N_{\varepsilon(x)}(x; X) = \bigcup_{x \in W} N_{\varepsilon(x)}(x; A)$$

which by construction is open in the restricted metric topology for A.■