# 11. The Hausdorff Condition

Hausdorff Separation Property in $(X, \mathcal{T})$

Given $x \neq y$ there are open subsets $U, V$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

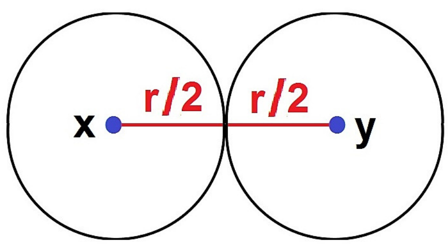
Example $(X, d)$ metric, $r = d(x,y)$.

$U = N_{r/2}(x)$, $V = N_{r/2}(y)$

If $z \in U \cap V$ then $r = d(x,y) \leq d(x,z) + d(z,y)$

$< \frac{r}{2} + \frac{r}{2} = r$

CONTRADICTION.




**Every pair of distinct points has a pair of disjoint open neighborhoods.**

<u>Example</u> Cofinite topology on infinite $X$ is not Hausdorff: $U, V$ open nonempty $\Rightarrow$ $U \cap V$ is infinite.

<u>Important properties in the exercises</u>:

EXER-
CISE #

(11.2) ① $X$ hausdorff $\Rightarrow$ every one point subset is closed

② $X$ Hausdorff $\Leftrightarrow$ diagonal $\Delta_X = \{(x_1, x_2) \in X \times X \mid$

(11.7) $x_1 = x_2\}$ is closed in $X \times X$

(11.10) ③ $f_0, f_1 : X \to Y$ continuous & $Y$ Hausdorff $\Rightarrow$ set of points in $X$ where $f_0(x) = f_1(x)$ is closed in $X$.

<u>Prop.</u> $A \subseteq X$ & $X$ Hausdorff $\Rightarrow A$ Hausdorff.

<u>Proof</u> $a_1 \neq a_2$ in $A$ There are open sets $U_i$ in $X$ such that $a_i \in U_i$ and $U_1 \cap U_2 = \emptyset$. Let $V_i = A \cap U_i$. Then $a_i \in V_i$ and $V_1 \cap V_2 = \emptyset$.

<u>Prop.</u> If $X$ and $Y$ are Hausdorff, so is $X \times Y$.

<u>Proof</u> Exercise 11.4. ∎

<u>Regular and normal spaces</u>

<u>Theorem</u> $(X, d)$ metric $E$ & $F$ disjoint closed subsets. Then there are open sets $U, V$ s.t. $E \subseteq U$, $F \subseteq V$ and $U \cap V = \emptyset$. (NORMAL SPACE)

<u>Proof.</u> Let $f(x) = \dfrac{d(x,E)}{d(x,E) + d(x,F)}$

The denominator is nonzero, for $E \cap F = \emptyset \Rightarrow$ $x \notin E$ or $x \notin F$, so either $d(x,E) > 0$ or $d(x,F) > 0$.

$f$ is continuous, $f = 0$ on $E$
$f = 1$ on $F$.

Take $U = f^{-1}[(-\infty, \frac{1}{2})]$, $V = f^{-1}[(\frac{1}{2}, +\infty)]$. ∎