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Urysohn's Lemma

As suggested in `math145Anotes11.pdf` a topological space $(X, \mathcal{T})$ is said to be **normal** if for each pair of disjoint closed subsets $E, F \subset X$ there are disjoint open sets U, V such that $E \subset U$ and $F \subset V$ (see the picture on the final page of this file). The following deep result has far-reaching consequences in further studies of topological spaces:

Theorem. *If X is a topological space then the following conditions are equivalent:*

(i) *The space X is normal.*

(ii) *For each pair of disjoint closed subsets $E, F \subset X$ there is a continuous function $f : X \to [0,1]$ such that $f|E = 0$ and $f|F = 1$.*

The implication $(ii) \Rightarrow (i)$ follows from the same considerations in the proof for metric spaces: We need only take $U = f^{-1}\left[\,[0, \frac{1}{2})\,\right]$ and $V = f^{-1}\left[\,(\frac{1}{2}, 1]\,\right]$. On the other hand the converse is highly nontrivial, and it is called **Urysohn's Lemma**. A detailed discussion of this result is given on pages 208–214 of Munkres, *Topology* (Second Edition). The *Wikipedia* article

`https://en.wikipedia.org/wiki/Pavel_Urysohn`

contains biographical information about the mathematician who discovered this result.