12. CONNECTED SPACES

Intermediate Value Property

f : interval in $\mathbb{R} \longrightarrow \mathbb{R}$ continuous

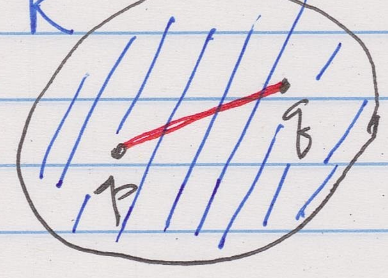
u, v in domain such that $f(u) < f(v)$,
c such that $f(u) < c < f(v)$. **THEN**
there is some w in the domain s.t. $f(w) = c$.

PROBLEM. Find simple but general criteria (for metric & top spaces) under which the conclusion is true.

EXAMPLE $\left\{\begin{matrix}\text{closed}\\ \text{open}\end{matrix}\right\}$ disk, $x^2+y^2 \left\{\begin{matrix}\leq\\ <\end{matrix}\right\} 1$

square $0 \leq x, y \leq 1$.

K



Examples of convex sets:
$p, q \in K \Rightarrow$ the closed segment $p + t(q-p) \in K$, $0 \leq t \leq 1$

"no dents or holes"

CLAIM The Intermediate value property is true for continuous functions on K.

First, let's verify the three sets are convex.

<u>Disks</u> $|p|, |q| \left\{ \begin{matrix} \leq \\ < \end{matrix} \right\} 1 \Longrightarrow$ $\boxed{\text{Euclidean norm!}}$

$$|p+t(q-p)| = |tq+(1-t)p| \leq$$

$$|t|\cdot|p| + |1-t|\,|q| \underset{t,\ 1-t\geq 0}{=\!=} t|p| + (1-t)|q|$$

$$\left\{ \begin{matrix} \leq \\ < \end{matrix} \right\} t\cdot 1 + (1-t)\cdot 1 = 1.$$

$\boxed{\text{Note that } p+t(q-p) = tq+(1-t)p}$

<u>Square</u> Write $p=(p_1, p_2)$
$q=(q_1, q_2)$

Both in $[0,1]\times[0,1]$. Then $tq + (1-t)p =$

$(tq_1 + (1-t)p_1, tq_2 + (1-t)p_2)$. Then

$0 \leq p_i, q_i \leq 1 \Rightarrow 0 \leq tq_i + (1-t)p_i \leq$
$0 \leq t \leq 1$ $\qquad t + (1-t) = 1.$ ∎

<u>Now, prove the claim</u> $f(w) < f(v)$

with $f: K \longrightarrow \mathbb{R}$ continuous.

Let $h(t) = f(tv + (1-t)w)$, $t \in [0,1]$.

Then h is continuous, $h(0) = f(u)$
$h(1) = f(v)$.

Therefore there is some $t^* \in (0,1)$ such that $h(t^*) = c$. But now

$$c = f(\underbrace{t^*v + (1-t^*)u}_{\text{lies in } K \text{ by convexity.}})$$

ISSUES 1. We have only shown that if intervals have the Intermediate Value Property, then so do convex sets

$\boxed{\text{See Sutherland Thm. 4.35}}$

2. We still want a simple criterion, expressible in terms of the topology on a set.

Def $(X, \mathcal{T})$ is connected if the only subsets U which are both open and closed (clopen!) are $\emptyset$ and X. Note U clopen in $X \Rightarrow$ so is $X - U$.

Example A discrete space with ≥ 2 points is not connected. ($\{x\}$ is clopen, all $x \in X$).

2. If $J \subseteq \mathbb{R}$ is such that $u < v$ in $J \Rightarrow (u,v) \subseteq J$, then J is connected.

<u>Proof of #2.</u> Suppose $\phi \neq U$, $J \neq U$ where $U \subseteq J$ is clopen. Then $V = J - U$ is also a nonempty proper clopen subset. Let $u \in U$ and $v \in V$. Without loss of generality, $u < v$. (if $u > v$, reverse roles of u, v and U, V in the discussion which follows).

By hypothesis, (u, v) and $[u, v] \subseteq J$.

Let $b^* = \text{l.u.b. } U \cap [u, v]$ (nonempty, for u lies in this set).

① Suppose $b^* < v$. Then $b^* = \text{l.u.b. } U \cap [u, v]$ and $[u, v] \subseteq J \Rightarrow (b^*, v] \subseteq V$. Since V is closed, we also have $[b^*, v] \subseteq V$ so that $b^* \in V$. If $b^* = v$ the same conclusion holds by assumption.

Hence $b^* > u$, since $u \in U$.

② <u>CLAIM:</u> There is a sequence of points $x_n \in U$ (n suff. large) such that

$$b^* - \frac{1}{n} < x_n < b^*$$

(Exists since b^* is a least upper bound)

Then $\lim_{n\to\infty} x_n = b^*$. Since $U \cap [u,v]$ is closed, we have $b^* \in U$.

① & ② contradict each other. The source of the problem is the assumption that J is a union of two nonempty disjoint clopen subsets. Hence this is false and J is connected. ∎

Converse Statement If $J \subseteq \mathbb{R}$ with $u < v$ in J and $u < x < v$ with $x \notin J$, then J is not connected.

Verification $(-\infty, x) \cup (x, +\infty) = \mathbb{R}$

disjoint open sets.

Hence J is the union of the nonempty disjoint subsets $\begin{matrix}(-\infty, x) \cap J \\ (x, +\infty) \cap J\end{matrix}$ (since $x \notin J$).

We can now prove an abstract form of the Intermediate Value Theorem.

<u>Sutherland, Prop. 2.11</u> $f: X \to Y$ continuous, X connected $\Longrightarrow f[X]$ connected.

<u>Proof.</u> Prove contrapositive: If $f[X]$ is not connected, then neither is X.

Suppose $W \subseteq f[X]$ is a nonempty, clopen proper subset. Write $W = f[X] \cap U$, U open in Y
$f[X] \cap E$, E closed in Y.

Then $f^{-1}[W] = f^{-1}[f[X] \cap U] = f^{-1}[f[X]] \cap f^{-1}[U]$
$= f^{-1}[U]$, open in X
also $= f^{-1}[E]$ closed in X.

Finally, since W is a nonempty proper subset of $f[X]$ we have $x_1, x_2 \in X$ such that $f(x_1) \in W$ and $f(x_2) \notin W$. Hence $f^{-1}[W]$ is a nonempty proper subset of X, so X is not connected. ∎

<u>Corollary</u> If $Y = \mathbb{R}$ and $f(x_1) < f(x_2)$, then $[f(x_1), f(x_2)] \subseteq f[X]$. ($X$ connected).

(<u>Intermediate Value Property</u>). ∎

the closed interval in the real line