Recognizing connected (sub) spaces

Sutherland, 12.16 $A_\alpha \subseteq X$ connected all α. Suppose for some α_0 we have $A_\alpha \cap A_{\alpha_0} \neq \emptyset$ all α. Then $\cup A_\alpha$ is connected.

We might as well replace X by $Y = \cup A_\alpha$.

Proof. Suppose Y is not connected, and let $W \subseteq Y$ be nonempty, clopen ($\Rightarrow Y - W$ clopen)

For each α, $W \cap A_\alpha$ is clopen in $A_\alpha \Rightarrow$ either $W \cap A_\alpha = \emptyset$ or $W \cap A_\alpha = A_\alpha$.

so $A_\alpha \subseteq Y - W$ so $A_\alpha \subseteq W$.

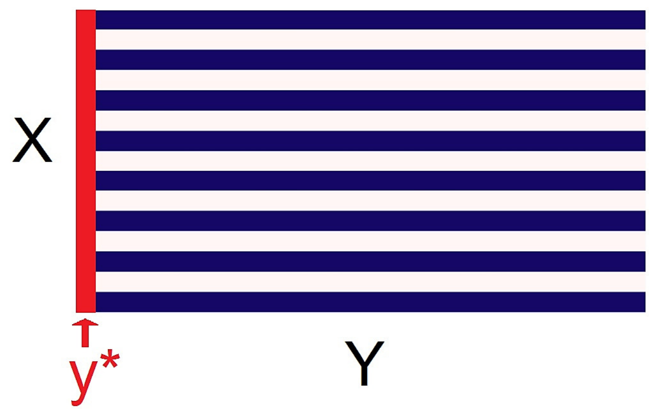
We might as well assume $A_{\alpha_0} \subseteq W$ (otherwise switch the roles of W and $Y - W$).

Since $A_\alpha \cap A_{\alpha_0} \neq \emptyset$ and $A_\alpha \cap A_{\alpha_0} \subseteq W$, we must have $W \supseteq A_\alpha$ ($A_\alpha \subseteq Y - W \Rightarrow A_\alpha \cap A_0 \cap W = \emptyset$).

Hence $\bigcup_\alpha A_\alpha \subseteq W$, so $W = Y$. ∎

Thm. 12.18 X, Y connected $\Rightarrow$ so is $X \times Y$.

Idea of proof:



First show that every horizontal strip of the form $\{x\} \times Y$ is connected, and likewise for some vertical strip of the form $X \times \{y^*\}$. Then apply the previous criterion for recognizing connected sets.

Proof. X is homeo to $X \times \{y_0\} \Rightarrow$ latter connected

Y is homeo to $\{x_0\} \times Y \Rightarrow$ $\boxed{\text{DITTO}}$

Hence for each $x \in X$ the set

$$A_x = (X \times \{y_0\}) \cup (\{x\} \times Y) \text{ is connected.}$$

THEY MEET AT (x, y_0).

We have $X \times \{y_0\} \subseteq A_x \cap A_{x'}$, so

$$X \times Y = \bigcup_{x \in X} A_x \text{ is connected.} \blacksquare$$

Cor. A. J_1, J_2 intervals in $\mathbb{R} \Rightarrow J_1 \times J_2$ conn.

Cor. B. $X_1, \dots, X_n$ conn. $\Rightarrow \prod_{i=1}^{n} X_i$ conn.

Prop. 12.19 A connected $\subseteq X$ and $A \subseteq B \subseteq \overline{A} \Rightarrow B$ connected.

Proof. Without loss of generality, we may assume $B = X$.

Suppose B is not connected, and let U be a nonempty, clopen, proper subset of B, so that $V = X - U$ is also a nonempty, clopen, proper subset.

We then have $A = (A \cap U) \cup (A \cap V)$ where $A \cap U$ and $A \cap V$ are clopen in A. Hence $\{A \cap U, A \cap V\} = \{A, \phi\}$ (in no particular order). Let's say (without loss of generality) that $A \cap U = A$, so that $A \subseteq U$. Now U is closed in B, so $B = \overline{A} \subseteq U$ and hence $B = U$. This is a contradiction. — The source is the assumption that B is not connected, so the latter must be false. Hence B is connected. ∎

<u>Cor.</u> If E is a subset of S^1, then $N_1(0; \mathbb{R}^2) \cup E$ is connected. — This means that $\mathbb{R}^2$ has many more connected subsets than $\mathbb{R}$ has (in fact, it's the same as the cardinal number of subsets of $\mathbb{R}^2$ itself). In fact the subsets in the corollary are all <u>convex</u>.

Path / Arcwise Connected Spaces

X — Given $x \neq y$ in X, can find

$\gamma : [a,b] \to X$ continuous so that $\gamma(a) = x$,

$\gamma(b) = y$. $\boxed{\text{Say } \gamma \text{ is a continuous curve joining } x \text{ to } y \text{ (or } x \text{ and } y\text{).}}$ ← OR ARC

Prop. ① Arcwise connected $\Rightarrow$ connected

② X, Y arcwise connected $\Rightarrow$ so is $X \times Y$.

Proof. ① Fix $p \in X$. If $x \in X$, let γ be a curve joining p to x, and let $\Gamma_x = $ image γ.

Then Γ_x is connected and $p \in \Gamma_x$ all x, so

$$X = \bigcup_{x \in X} \{x\} \subseteq \bigcup_{x \in X} \Gamma_x \subseteq X \Rightarrow X = \bigcup_{x \in X} \Gamma_x .$$

By a previous result, RHS is connected $\Rightarrow$ so is X. ∎

② Let (x_1, y_1) & $(x_2, y_2) \in X \times Y$.

$$\text{Let } \left\{ \begin{array}{l} \gamma_X : [a_X, b_X] \to X \\ \gamma_Y : [a_Y, b_Y] \to Y \end{array} \right\}$$

join $\left\{ \begin{array}{l} x_1 \text{ to } x_2 \\ y_1 \text{ to } y_2 \end{array} \right\}$, and let h_X, h_Y be linear $[0,1] \to \begin{array}{l} [a_X, b_X] \\ [a_Y, b_Y] \end{array}$

functions with $\begin{cases} h_X(0) = a_X, & h_X(1) = b_X \\ h_Y(0) = a_Y, & h_Y(1) = b_Y \end{cases}$.

Then
$$\begin{tikzcd}[0,1] \arrow[r, "(h_X, h_Y)"] \arrow[dr, "\beta"'] & {[a_X, b_X] \times [a_Y, b_Y]} \arrow[d, "\gamma_X \times \gamma_Y"] \\ & X \times Y \end{tikzcd}$$

β joins (x_1, y_1) to (x_2, y_2). $\blacksquare$

<u>Prop. 12.25.</u> If U is nonempty and open in $\mathbb{R}^n$ $(n \geq 1)$ with U connected, then U is arcwise connected.

<u>Proof.</u> Given any space X, write $x \sim y \Leftrightarrow$ there is a cont curve $\gamma : [a,b] \to X$ joining x and y. <u>CLAIM</u>: This is an equivalence relation.

$\boxed{x \sim x}$ Take γ = constant curve at x.

$\boxed{x \sim y \Rightarrow y \sim x}$ Given γ as above, define

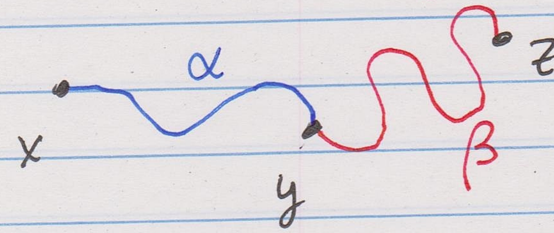
$$\gamma^* : [-b, -a] \xrightarrow{(-1)} [a, b] \xrightarrow{\gamma} X,$$

so $\gamma(-b) = y$ & $\gamma(-a) = x$.

$\boxed{x \sim y \text{ \& } y \sim z \Rightarrow x \sim z}$ Given $\begin{cases} \alpha : [a,b] \to X \\ \beta : [c,d] \to X \end{cases}$

$\alpha(a) = x, \ \alpha(b) = y = \beta(c), \ \beta(d) = z.$

CONCATENATE (string together) THE CURVES.



$\gamma = \alpha + \beta$

$\gamma : [a, b+d-c] \longrightarrow X \quad \gamma(t) = \alpha(t)$ if $t \le b$

$\gamma(t) = \beta(c+t-b)$ if $t \ge b$. These fit together

because $\underset{t=b}{\alpha(b)} = y = \beta(c + \overset{t=b}{\cancel{b-b}})$.

Note that $\gamma(b+d-c) = \beta(c+(b+d-c)-b) = \beta(d) = z$

The equivalence classes are open, for if

in $\mathbb{R}^n$ $N_\varepsilon(p) \subseteq U$, then $N_\varepsilon(p) \subseteq$ equivalence class of p.

Equivalence classes are pairwise disjoint, so if there is more than one equivalence class, then each equivalence class is an open subset, and likewise for its complement (which is a union of eq. classes). Since U is conn., only one eq. class, and this implies U is arcwise conn. ∎