Application – Showing Spaces are not homeomorphic

Prop. $f: X \to Y$ homeomorphism & $x \in X$ $\Longrightarrow$ f maps $X - \{x\}$ homeomorphically to $Y - \{f(x)\}$.

Proof. Let U be open in $X - \{x\}$, so $U = W \cap (X - \{x\})$ where W is open in X. Then $f[U] = f[W] \cap f[X - \{x\}] = f[W] \cap Y - \{f(x)\}$, where $f[W]$ is open in Y since f is a homeo; hence $f[U]$ is open in $Y - \{f(x)\}$. Conversely, if V is open in $Y - \{f(x)\}$, then $V = W \cap (Y - \{f(x)\})$ where W is open in Y. Then $f^{-1}[V] = f^{-1}[W] \cap (X - \{x\})$; by continuity $f^{-1}[W]$ is open in X, so $f^{-1}[V]$ is open in $X - \{x\}$. Hence $f_0 : X - \{x\} \to Y - \{f(x)\}$, with $f_0(t) = f(t)$, is a homeomorphism. ∎

Cor. Same conclusion if $\{x\}$ is replaced by $\{x_1, \ldots, x_k\}$ etc. [induction]

Distinguishing interval types

$[a, b]$ — There are exactly two points in X such that $X - \{x\}$ is connected.

(a, b) — $X - \{x\}$ is never connected.

$[a, b)$ — There is a unique point in X such that $X - \{x\}$ is connected.

S^1 — For each x, $X - \{x\}$ is conn.

PROOF. $x = (\cos 2\pi s_0, \sin 2\pi s_0)$ some s_0.

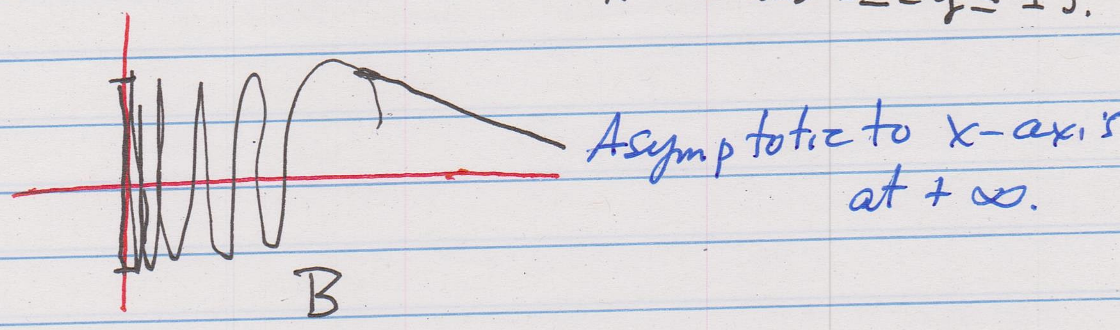
Let $\gamma(t) = (\cos 2\pi(s_0 + t), \sin 2\pi(s_0 + t))$ $0 < t < 1$. Then image $\gamma = S^1 - \{x\}$, so the latter is connected.

NO PROOF. S^1 - two pts. is not connected, but $\mathbb{R}^2$ - two pts is connected.

Example

let $B = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0$ and $y = \sin \frac{1}{x}$ or

$x = 0$ and $-1 \leq y \leq 1\}$.



Then B is connected but not arcwise conn.
(no continuous curves in B joining $(0,0)$ to points with $x > 0$).

See the second example on p. 65 of graduate-level-classnotes.pdf and further discussion of it on p. 66 for an explanation.

Addendum. Monotonic, Continuous functions of one real variable

The concepts and results of Chapter 12 allow us to prove the following basic theorems which has been stated and used in examples and homework without proof.

monotonic = either increasing or decreasing

THEOREM. Let $J \subseteq \mathbb{R}$ be a/an $\left\{\begin{matrix}\text{open}\\ \text{half open}\\ \text{closed}\end{matrix}\right\}$ interval, and let $f : J \rightarrow \mathbb{R}$ be continuous and strictly (in/de)-creasing. Then $f[J] = K$ is an interval of the same type as J, and there is a continuous, strictly (in/de)-creasing function $g : K \rightarrow \mathbb{R}$ which is inverse to f. Specifically, $g[K] = J$ and we have $g(f(x)) = x$ $(x \in J)$ $f(g(y)) = y$ $(y \in K)$. ∎

A proof of this result is given in the file intro2topA-12a.pdf in the course directory.