13. Compact spaces

Extreme Value Property

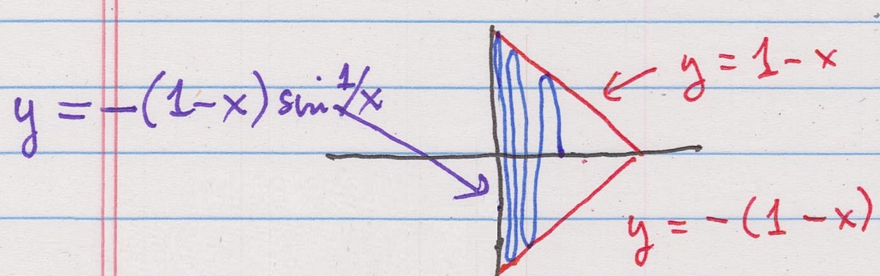
$f : [a,b] \longrightarrow \mathbb{R}$ continuous $\Longrightarrow$

f takes a maximum and minimum value on $[a,b]$.

Examples ① $f(x) = x$ $\mathbb{R} \to \mathbb{R}$, no upper or lower bounds on the values $f(x)$.

② $f(x) = \dfrac{x}{1+|x|}$ $\mathbb{R} \longrightarrow \mathbb{R}$ l.u.b of all $f(x)$ is $+1$, g.l.b. is -1, but $f(x)$ is never ± 1.

③ $f(x) = -(1-x)\sin\frac{1}{x}$ on $(0, \frac{1}{\pi})$.



l.u.b $f(x) = +1$

g.l.b. $f(x) = -1$

but $f(x)$ is never ± 1.

Problem. Find general criteria under which the conclusion is true $\boxed{\text{on metric + top. spaces}}$

Let's see how one proves the Extreme Value Property

Heine-Borel-Lebesgue Theorem

Let $[a, b]$ be a closed interval, and suppose that for each $x \in [a, b]$ we are given an open interval $(x - \delta_x, x + \delta_x)$. Then there is a finite number of points $x_1, \dots, x_m$ such that $[a, b] = \bigcup_{i=1}^{m} (x_i - \delta(x_i), x_i + \delta(x_i))$.

Proof. Let $J_x = (x - \delta_x, x + \delta_x)$, and say $y \in [a, b]$ is accessible if $[a, y]$ is contained in a union of finitely many J_x's.

CLAIM $y^* =$ L.U.B. of accessible points $\Longrightarrow$ $y^* = b$ (note that a is accessible). THIS PROVES THM.

① y^* is accessible. By def of L.U.B., there is an accessible point x^* in $(y^* - \delta_{y^*}, y^*]$. Regardless of whether $x^* < y^*$ or $x^* = y^*$, we see that y^* is accessible (add J_{y^*} to the finite collection of intervals).

② If $y^* < b$, then $y^* < z \le b \Rightarrow z$ is not accessible because y^* is the L.U.B of accessible points.

③ On the other hand, every point in $J_{y^*} \cap [a,b]$ is also accessible. CONTRADICTION

The source of the problem is the hypothesis that $y^* < b$. — Hence y^* must be equal to b. ∎

Corollary 1. f is bounded on $[a,b]$.

Proof Let $\varepsilon > 0$, and take J_x's so that $t \in (x-\delta_x, x+\delta_x) \Rightarrow |f(t)-f(x)| < \varepsilon$.

By HBL, $[a,b] \subseteq J_{x_1} \cup \ldots \cup J_{x_m}$.

Hence $f(x) \le \max\{f(x_i)+\varepsilon\}$

$f(x) \ge \min\{f(x_i)-\varepsilon\}$. ∎

Corollary 2. f takes maximum and minimum values.

$$m = \underset{x}{GLB}\, f(x) \le \underset{x}{LUB}\, f(x) = M.$$

We might as well assume $m > 0$; replace $f(x)$ by $g(x) = f(x) + C$ where $m + C > 0$.

① Suppose $f(x) < M$ always. Then

$\frac{1}{M - f(x)}$ is continuous but unbounded. ⊛

② Suppose $m < f(x)$ always. Then

$\frac{1}{f(x) - m}$ is continuous but unbounded ⊛

⊛ In each case this is true because

$\left\{\begin{matrix} M - f(x) \\ f(x) - m \end{matrix}\right\} > 0$ but for each $K > 0$

we can find x such that $\left\{\begin{matrix} M - f(x) \\ f(x) - m \end{matrix}\right\} < \frac{1}{K}$.

Hence we can find x s.t. $\frac{1}{M - f(x)}$ or $\frac{1}{f(x) - m} > K$.

respectively

In both cases the first corollary yields a contradiction. The source is the assumption that $\left\{\begin{matrix} f(x) < M \\ f(x) > m \end{matrix}\right\}$ always, so this is false

and $f(x)$ can be chosen to be either M or m. ■