Putting Things into a more general setting

Want a comparable result with $[a,b]$ replaced by a closed bounded subset of $\mathbb{R}^n$; but even more general than that.

<u>Def.</u> A topological space X is <u>compact</u> if for every family of open subsets $\mathcal{U} = \{U_\alpha\}$ such that $X = \bigcup_\alpha U_\alpha$, there is a finite subcollection $\{U_{\alpha_1}, \dots, U_{\alpha_k}\}$ such that $X = U_{\alpha_1} \cup \dots \cup U_{\alpha_k}$.

Say $\mathcal{U}$ = open covering, $\{U_{\alpha_1}, \dots, U_{\alpha_k}\}$ finite subcovering

<u>Example</u> Heine-Borel-Lebesgue $\Rightarrow$ $[a,b]$ is compact.

We can now modify the proof of the Extreme Value Property so that it is valid if X is a compact space.

<u>NEXT</u> Which subsets of $\mathbb{R}^n$ are compact?

Sutherland, Prop. 13.10 (X, d) metric and $A \subseteq X$ is compact $\Rightarrow$ A is bounded.

Proof. Without loss of generality, $A = X$. Fix $\varepsilon > 0$ and take the open covering $\mathcal{U}$ whose elements are the sets $N_\varepsilon(x)$ for $x \in X$. Let $N_\varepsilon(x_1), \ldots, N_\varepsilon(x_k)$ be a finite sub-covering, and let $x_0 \in X$. Let $D = \max$ of $d(x_0, x_i)$. Then $y \in X \Rightarrow y \in N_\varepsilon(x_i)$ some i $\Rightarrow d(x_0, y) \leq d(x_0, x_i) + d(x_i, y) <$

$$M + \varepsilon. \blacksquare$$

Sutherland, Prop 13.12 $(X, \mathcal{T})$ Hausdorff and $A \subseteq X$ compact $\Rightarrow$ A closed in X.

So $A \subseteq \mathbb{R}^n$ compact $\Rightarrow$ closed, bdd

Proof. Show $X - A$ is open in X. Let $y \in X - A$, $z \in A$. Then there are disjoint open subsets $U_{y,z}$ & $V_{y,z}$ containing y and z respectively.

The sets $\{V_{y,z} \cap A\}$ form an open covering of A. Take a finite subcovering $V_{y,z_1} \dots V_{y,z_k}$ and let $U_y = U_{y,z_1} \cap \dots \cap U_{y,z_k}$. * Then $y \in U_y$ and

V_{y,z_j} abbreviated

$$U_y \cap A \subseteq U_y \cap (\cup V_j) =$$

$$\cup (U_y \cap V_j) \subseteq \cup (U_{y,z_j} \cap V_{y,z_j}) =$$

$\cup$ empty sets $= \emptyset$. Hence

$X - A = \bigcup_{y \notin A} U_y$ must be open in X. ∎

*

See the drawing on p. 131 of Sutherland

To finish we must show that if $A \subseteq \mathbb{R}^n$ is closed & bounded, then it is compact. This can be done abstractly as follows.

<u>Sutherland, Prop. 13.20</u> If A is closed in X and X is compact, then A is also compact.

Proof. Let $\{U_\alpha\}$ be an open covering of A. For each U_α, let $V_\alpha \subseteq X$ be open such that $U_\alpha = V_\alpha \cap A$. Let $\mathfrak{V}$ = all V_α's plus $X-A$. Then $\mathfrak{V}$ is an open covering of $X \Rightarrow$ there is a finite subcovering $V_{\alpha_1}, \ldots, V_{\alpha_n}, X-A$.

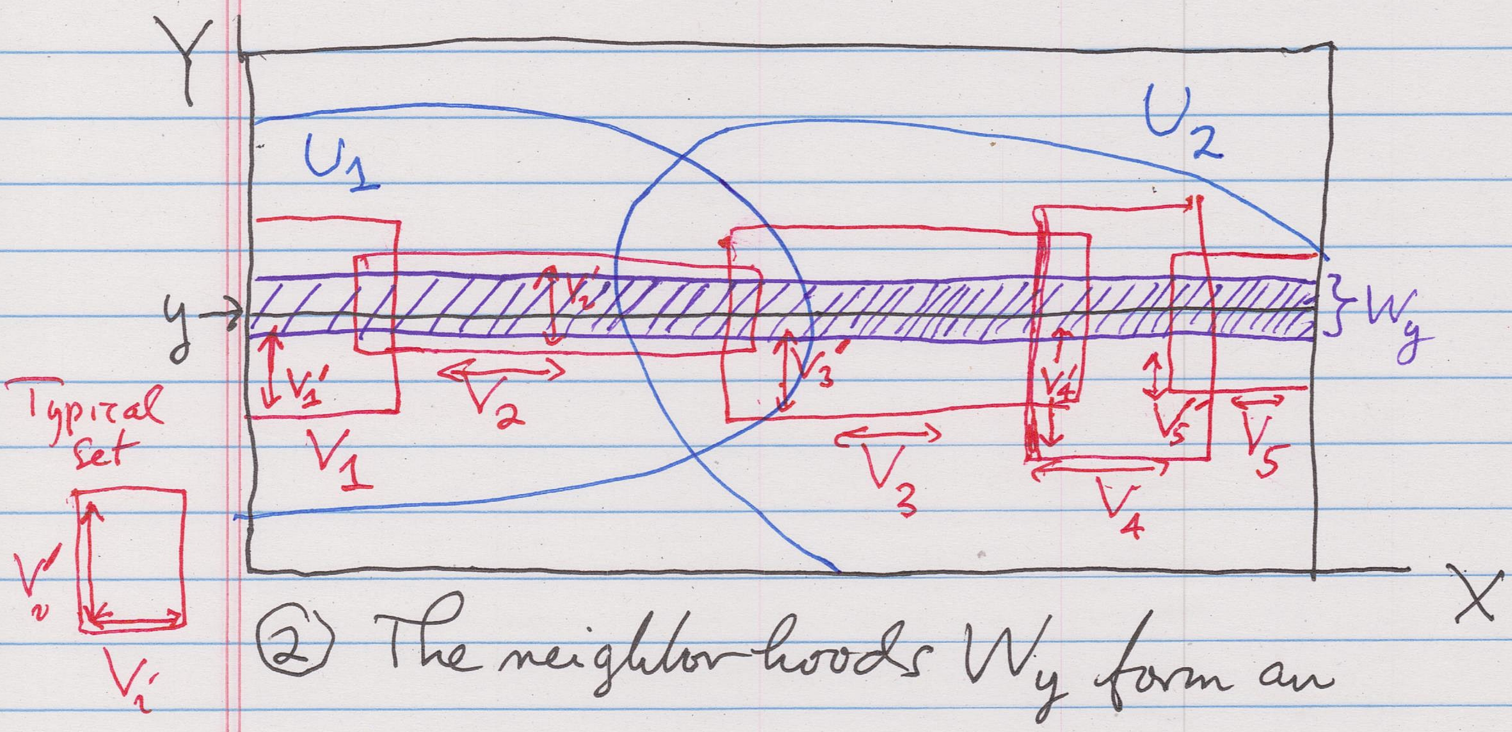
Then $U_{\alpha_1} \cup \ldots \cup U_{\alpha_n} = (V_{\alpha_1} \cap A) \cap \ldots \cap (V_{\alpha_n} \cap A)$
$= (\cup V_{\alpha_i}) \cap A$ Now $\cup V_{\alpha_i} \supseteq A$, so the last expression equals A. ■

Sutherland, Thm. 13.21 X, Y compact $\Longrightarrow X \times Y$ compact.

Proof Let $\mathfrak{U}$ = open covering of $X \times Y$ with open sets U_α.

1 Let $y \in Y$. Then $X \times \{y\} \cong X$ is compact $\Rightarrow$ there is a finite subcovering $U_1, \ldots, U_n$ of $X \times \{y\}$. CLAIM: $U_1, \ldots, U_n$ covers a thickened slice $X \times W_y$ for some open nbhd W_y of y.

Given $(x,y) \in X \times \{y\}$, Pick U_i such that $(x,y) \in U_i$, then take V_x open in X, V_x' open in Y such that $(x,y) \in V_x \times V_x' \subseteq U_i$. The sets $\{V_x\}$ form an open covering of X. Take a finite subcovering $\{V_{x_j}\}$. If $W_y = \bigcap_j V_{x_j}'$, then $X \times W_y = \bigcup V_{x_j} \times W_y$. But each of these sets is contained in some U_i, so $X \times W_y \subseteq \bigcup_i U_i$.



② The neighborhoods W_y form an open covering of X.

Take a finite subcovering $W_{y_1}, \dots W_{y_s}$

Hence $X \times Y = \bigcup_k X \times W_{y_k}$

For each k we have a finite subcollection $\mathcal{U}_k$ whose union contains $X \times W_k$, so $\bigcup_k \mathcal{U}_k \subseteq \mathcal{U}$ is a finite subcovering of $X \times Y$. ∎

<u>Summary</u> $K \subseteq \mathbb{R}^n$ closed $\Leftrightarrow$ closed and bounded.

1. K compact $\Rightarrow$ closed, bounded.
2. K bounded $\Rightarrow$ $K \subseteq [a_1, b_1] \times \dots \times [a_n, b_n]$.

   <u>Proof</u> Suppose $v \in K \Rightarrow |v|_2 \le M$. If $v = (v_1, \dots, v_n)$, then $|v_i| \le |v|_2 \le M$ so $K \subseteq [-M, M]^n$.
3. $[-M, M]^n$ is compact.
4. K closed, $K \subseteq [-M, M]^n \Rightarrow K$ is a closed subset of a compact set $\Rightarrow$ K is compact. ∎

NOTE. We can combine the results from Chapters 12 and 13 to conclude that if $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, then its image is a closed interval (it has a maximum and minimum, and it contains all intermediate points). However, if we replace $\mathbb{R}$ with $\mathbb{R}^n$ for $n \geq 2$, then there are many more possibilities, including some that are surprising. For example, in $\mathbb{R}^2$ the image can be the unit square $[0,1] \times [0,1]$.

Additional information about such examples (and further references) can be found in

intro2topA-13.pdf.

CAUTION: More generally, the preceding discussion shows that a compact subset of a metric space is closed and bounded. However, a closed bounded subset of an arbitrary metric space is not necessarily compact. For example, consider the case X = A = (0,1).