Here are some details. We shall not try to prove that sequentially compact $\Rightarrow$ compact in this course.

Proof that Sequentially Compact $\Leftrightarrow$ Limit Point Compact for Metric Spaces

($\Rightarrow$) Assume X is ~~sequentially~~ limit point compact, and let $\{x_n\}$ be a sequence in X. Let $A = \{x_1, x_2, \dots\}$. If A is finite, then $A = \{y_1, \dots, y_k\}$ and some $y_j = x_n$ for ∞ many n. Hence there is a subsequence $\{x_{n_i} = y_j\}$ for some fixed y_j, and this sequence converges to y_j. — Now assume A is infinite. Then there is some limit point $b \in L(A)$. Given $k > 0$, there is some $x_{n(k)}$ such that $d(x_{n(k)}, b) < \frac{1}{2^k}$. In fact, since $(N_{1/2^k}(b) - \{b\}) \cap A$ is infinite, one can find such a sequence inductively such that $x_{n(k)} \neq x_{n(l)}$ for $l < k$ (only finitely many of the latter).

Clearly $b = \lim_{k \to \infty} x_{n(k)}$. $\blacksquare$

($\Leftarrow$) Assume X is sequentially compact, and let $A \subseteq X$ be infinite. Then there $\exists$ an infinite sequence of distinct points $\{a_n\}$ in A, and this sequence has a convergent subsequence $\{a_{n_k}\}$. Let b be the limit of this subsequence. <u>CLAIM</u>: $b \in L(A)$.

Let $\varepsilon > 0$; we need to show that $(N_\varepsilon(b) - \{b\}) \cap A \neq \emptyset$. In fact, for all k sufficiently large $a_{n_k} \in N_\varepsilon(b) \cap A$. If $b \neq a_{n_\ell}$ for all ℓ, then $(N_\varepsilon(b) - \{b\}) \cap A$ will be nonempty. Also, if $b = a_{n(\ell)}$ for some ℓ, then ℓ such that $m \geq \ell \Rightarrow a_{n(m)} \in N_\varepsilon(b)$, then $a_{n(\ell+1)} \neq \underset{(=b)}{a_{n(\ell)}}$ and $a_{n(\ell+1)} \in (N_\varepsilon(b) - \{b\}) \cap A$ and hence the latter is still nonempty. Therefore $b \in L(A)$ as required. $\blacksquare$

Proof that a compact space is limit point compact

(No assumptions about metrics)

X compact, $A \subseteq X$ infinite subset.

Suppose $L(A) = \phi$. Then A is closed.

<u>CLAIM</u> If $a \in A$, then $\{a\} \subseteq A$ is open in the subspace topology. — To see this, note that $a \notin L(A) \Rightarrow$ there is some V open in X so that $a \in V$ and $(V - \{a\}) \cap A = \phi$. In other words, $A \cap V = \{a\}$, so that $\{a\}$ is open in A.

Hence $A \subseteq X$ is a closed, discrete subset of X, and hence

1. by closed, A is compact
2. by discrete, A is not compact

Contradiction

The source of the contradiction is the assumption that $L(A) = \phi$, and hence we must have $L(A) \neq \phi$. ∎

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