Closed Bounded Subsets of Normed Vector Spaces

The goal is to give an example as follows:

EXAMPLE. There is a closed bounded subset A in a normed vector space V such that

(i) A satisfies a strong form of closure called completeness,

(ii) There is a continuous function $f : A \longrightarrow \mathbb{R}$ which is unbounded. FROM ABOVE OR BELOW

In contract, if $V = \mathbb{R}^n$ with any of the norms $|v|_p$ ($p = 1, 2, \infty$) and $A \subseteq V$ is closed + bounded, then every continuous function $f : A \longrightarrow \mathbb{R}$ attains a maximum and a minimum value (see math145Anotes13.pdf for more on the latter).

In fact, A is isometric to $\mathbb{N}$ with the usual discrete metric.

<u>Preliminary facts</u> (i) Every discrete metric space is complete; i.e., every Cauchy sequence (Sutherland, p. 68) converges.

(ii) If X is a metric space and $A \subseteq X$ is complete, then A is closed in X (see Sutherland, Prop. 17.7).

<u>Proof of (i)</u> By the definition of a Cauchy sequence, there is some N such that $m, n \geq N$ $\Rightarrow d(a_n, a_m) < 1$ (assuming $\{a_k\}$ is Cauchy).

<u>CLAIM</u> If $k \geq N$, then $a_k = a_N$.

This is true since if $d(p,q) < 1$ in a discrete metric space, then $p = q$.

It follows that $\lim_{n \to \infty} a_n = a_N$. (Justify this!?!)

Construction of an example:

$\mathbb{R}^\infty$ = all sequences $(x_1, x_2, \dots)$ such that $x_i = 0$ for all but finitely many i,

$\|v\|_\infty$ on $\mathbb{R}^\infty$ = $\max |v_i|$ where $v = (v_1, \dots)$

(the maximum exists because only finitely many v_i's are nonzero).

Check this is a norm (left as an exercise).

Let $e_j = (0, \dots, 1, 0, \dots)$

$\uparrow$ 1 in jth position, zeros elsewhere.

Then $i \neq j \Rightarrow d_\infty(e_i, e_j) = 1$.

Let A = set of vectors $\{e_1, e_2, \dots\}$.

Since every function $f: A \to \mathbb{R} \leftarrow$ usual topology

is continuous, if we let

$f(e_j) = (-1)^j \, j \, e_j$

then f is continuous.

Then f has neither a maximum nor a minimum. In fact, there are no (finite) constants L, K such that $f(x) \geq L$ or $f(x) \leq K$. ∎

<u>Note</u> If we compose f with the map φ $\mathbb{R} \longrightarrow (-1, 1)$ sending t to $\dfrac{t}{1+|t|}$, then the composite $g = \varphi \circ f$ is bounded, but it does not attain either a maximum or a minimum value. (Note that $\text{l.u.b}\ g(t) = +1$ and $\text{g.l.b}\ g(t) = -1$).