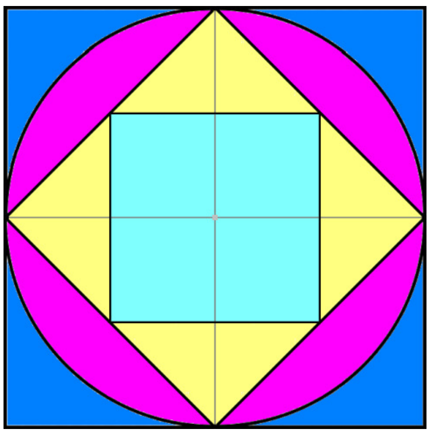
Topologically equivalent metrics on a product of two metric spaces

As noted in Chapter 5 of Sutherland (particularly Exercises 5.15 and 5.16 on page 58), if **X** and **Y** are metric spaces then there are three product metrics d_p — where **p = 1, 2** or **∞** — on the Cartesian product **X × Y** which yield the same underlying topological structure. The basic underlying reason for this is the string of inequalities $d_\infty \leq d_2 \leq d_1 \leq 2 \cdot d_\infty$. These inequalities can be restated in terms of the d_p open disks as follows:

- *For every real number* $r > 0$, *the open* d_2 *disk of radius* r *is contained in the open* d_∞ *disk of radius* r.
- *For every real number* $r > 0$, *the open* d_1 *disk of radius* r *is contained in the open* d_2 *disk of radius* r.
- *For every real number* $r > 0$, *the open* d_∞ *disk of radius* $\frac{1}{2}r$ *is contained in the open* d_1 *disk of radius* r.

Here is a picture which illustrates these inequalities. The light blue square in the middle is the open d_∞ disk of radius $\frac{1}{2}r$, the union of this square with the yellow triangular regions is the open d_1 disk of radius r, the union of these with the magenta circle segments is the open d_2 disk of radius r, and the large square is the open d_∞ disk of radius r.



These three metrics turn out to be part of a continuous family of metrics d_p for **X × Y** which are defined for all p satisfying $1 \leq p \leq \infty$; this follows from the ***Minkowski inequality***

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p}$$

which holds for all $p \geq 1$ (see **http://en.wikipedia.org/wiki/Minkowski_inequality** for further information and a proof).