Solutions for Quiz 1, Winter 2019

1. Let (X, d) be a metric space, and let $f : X \to \mathbb{R}$ be a continuous function. Prove that the function

$$d'(x, y) \quad = \quad d(x, y) \ + \ |f(x) - f(y)|$$

also defines a metric on X.

SOLUTION

We shall verify the defining properties of a metric in order:

The function d' is a sum of two nonnegative functions and hence is nonnegative. Furthermore, if $d'(x, y) = 0$ then the two summands $d(x, y)$ and $|f(x) - f(y)|$ must

Next, we need to verify that $d'(x, y) = d'(y, x)$ for all x and y. This follows because the corresponding property holds for the summands. Specifically, since d is a metric we have $d(x, y) = d(y, x)$, and since the absolute value satisfies $|a| = |-a|$ we have $|f(x) - f(y)| = |f(y) - f(x)|$. If we add these equations it follows that $d(x, y) = d(y, x)$.

Finally, we need to verify the Triangle Inequality. Examine the summands separately. The first one satisfies $d(x, z) \leq d(x, y) + d(y, z)$ because it is a metric, and the second one satisfies

$$|f(x) - f(z)| \quad = \quad |(f(x) - f(y)) + (f(y) - f(z))| \quad \leq \quad |f(x) - f(y)| \ + \ |f(y) - f(z)|$$

by the Triangle Inequality for the usual metric on the real line. If we add the inequalities $a \leq c$ and $b \leq d$, we obtain the new inequality $a + b \leq b + d$, and if we now apply this to the preceding inequalities, we obtain the Triangle Inequality for d'.■

Note that we did not use the continuity assumption on f.

2. Let $r(x, y) = |x^2 - y^2|$, where x and y are real numbers. Prove that r defines a metric on the set $[0, \infty)$ of nonnegative reals but does not define a metric on the set $\mathbb{R}$ of all real numbers.

SOLUTION

We shall first show that the function r defines a metric on the positive real numbers. Since absolute values are nonnegative, the quantity $r(x, y)$ is always nonnegative. Furthermore, it is zero if and only if $0 = |x^2 - y^2|$, which is true if and only if $x^2 = y^2$. Over the positive integers this happens if and only if $x = y$.

Next, we need to verify that $r(x,y) = r(y,x)$ for all x and y. As before, since the absolute value satisfies $|a| = |-a|$ we have $|x^2 - y^2| = |y^2 - x^2|$, which means that that $r(x,y) = r(y,x)$.

The Triangle Inequality now follows from the following equations and inequalities:

$$r(x,z) \;=\; |x^2 - z^2| \;=\; |(x^2 - y^2) + (y^2 - z^2)| \;\leq\; |x^2 - y^2| + |y^2 - z^2| \;=\; r(x,y) + r(y,z)$$

and therefore r defines a metric on the positive real numbers.

However, r does not define a metric on the set of all real numbers. If $y = -x$ is nonzero then $r(x,y) = 0$ but $x \neq y$, so over the reals the function r fails to satisfy one of the defining conditions for a metric on the set of all real numbers.∎