Solutions to 2020 problems

1. The union is the discrete topology because it contains all other topologies, and the intersection is the indiscrete topology because all other topologies contain it. $\blacksquare$

2. (a) Follow the hint. By assumption the statement is true if $k=0$. Assume it is true for $k-1 \geq 0$, so that every subset with $n-k+1 \leq n$ points is open. Suppose now that A has $n-k \leq n-1$ elements. Then one can find $u \neq v$ so $u, v \notin A$. Furthermore,

$$A = (A \cup \{u\}) \cap (A \cup \{v\}).$$

By induction, ~~Therefore~~ $A \cup \{u\}$ and $A \cup \{v\}$ are open, so their intersection, which is A, is also open.

This proves the induction step. — It follows that every one point subset is open. But $B \subseteq X$ $\Rightarrow B = \bigcup_{b \in B} \{b\}$, so every $B \subseteq X$ is open too. $\blacksquare$

(b) If we substitute "closed" for "open" in (a) we see that every FINITE subset is closed. $\blacksquare$

3. Every open subset V in Y has the form $\bigcup_\alpha (S_{\alpha_1} \cap \dots \cap S_{\alpha_k})$, so $f^{-1}[V]$ is just $\bigcup_\alpha (f^{-1}[S_{\alpha_1}] \cap \dots \cap f^{-1}[S_{\alpha_k}])$. Now each $f^{-1}[S_{\beta_j}]$ is assumed to be open, so the set $\bigcup_\alpha$(ETC.) must also be open, so f is continuous. ∎

4. <u>No</u>. This fails if (X, d) is a discrete metric space.

5. Since $f(p) = q$ we have $q = mp + b$. To find m, note that $|y - p| = \frac{1}{2} r \Rightarrow$ $|f(y) - f(p)| = \frac{1}{2} s$. But should be $m(y-p)$, so this suggests $m = \frac{s}{r}$ and hence $q = \frac{s}{r} p + b$, or $b = q - \frac{s}{r} p$. Hence we obtain the formula $f(y) = \frac{s}{r}(y - p) + q$. This is clearly continuous, but we also must

check that it has a continuous inverse. We need to show that ~~if~~ $x = f(y)$ can be solved uniquely for y, and this is true for every x. But we have

$$x = \frac{s}{r}(y-p) + q$$

so that

$(x-q) = \frac{s}{r}(y-p)$ or equivalently

$\frac{r}{s}(x-q) = y-p$, or $y = \frac{r}{s}(x-q)+p$. ■

2019 problems

1. A subset $V \subseteq A$ is open in the subspace topology $\Leftrightarrow$ it has the form $V = U \cap A$, where U is open in the indiscrete topology. The only possibilities for U are ϕ and X, so the only open subsets in the subspace topology are $\phi \cap A = \phi$ and $X \cap A = A$. ■

2. Every open subset of $\mathbb{R}$ is a union of sets having the form $(x-r, x+r)$ where $x \in \mathbb{R}$ and $r > 0$. Each set of this type is an intersection $(-\infty, x+r) \cap (x-r, \infty)$, so every open subset of $\mathbb{R}$ is a union of finite intersections of ~~inter~~ the sets described. Conversely, since these sets are open in $\mathbb{R}$, every union of finite intersections of such sets is open in $\mathbb{R}$. ∎

3. If $U \neq \emptyset$ is open then $X - U$ is finite. Let $y \in X - U$, and suppose $y \in V$, where V is open (hence infinite since all nonempty open sets are!). Then $y \in L(U)$ because the intersection $(V - \{y\}) \cap U$ is an intersection of two nonempty open sets & must be nonempty (its complement is finite). Therefore

$$\overline{U} = U \cup L(U) \subseteq U \cup (X - U) = X. \quad ∎$$

4. Sometimes T, sometimes F.

If X is finite this is true, for the conditions on both open & closed subsets are (1) ϕ, X in family, (2) the family is closed under finite unions + intersections since every $\left\{\begin{matrix}\text{union}\\ \text{intersection}\end{matrix}\right\}$ is expressible as a finite one.

If $X = \mathbb{R}$ this is false because the family of closed subsets does not satisfy F_n closed $(n \geq 0) \Rightarrow \cup F_n$ closed. ∎

5. False except in extreme cases (say $\mathcal{U}_1$ discrete).

Let X have topologies $X = \{1, 2, 3\}$

$\mathcal{U}_1 = \{X, \phi, \{1\}\}$, $\mathcal{U}_2 = \{X, \phi, \{2\}\}$.

Then $\mathcal{U}_1 \cup \mathcal{U}_2$ is not closed under taking unions. ∎

6. True if X, Y finite, but otherwise false. Say $|Y| = \infty$, $X = \phi$. Then each slice $\{p\} \times Y$ is infinite and closed.

<u>2017 problems</u>

1-6 are straight from the notes. See the files

quiz2review.2017.pdf

quiz2review2.2017.pdf

for the others.