CONVERSELY

Theorem (X, d) sequentially compact metric space $\Rightarrow$ X is compact.

Step 1 Under the hypotheses, for each $\varepsilon > 0$ there exists a finite collection of sets $N_\varepsilon(x_i)$ which cover X.

Proof of Step 1 Suppose for some $\varepsilon > 0$ there is no such finite covering. Define a sequence $\{x_n\}$ in X s.t. $d(x_i, x_j) \geq \varepsilon$ for each $i \neq j$.

Pick any x_1. Given $x_1, \ldots, x_{n-1}$ with the property $d(x_i, x_j) \geq \varepsilon$, find x_n by noting that

$$\bigcup_{i=1}^{n-1} N_\varepsilon(x_i) \subsetneq X$$ and picking $x_n \in X -$ union.

Let $S = \{x_1, x_2, \ldots\}$, an infinite set.

By sequential compactness, $L(S) \neq \phi$, so let $y \in L(S)$. Take $\{y_n\}$ in S such that the points y_n are distinct and $y_n \to y$.

Then $n \geq N \Rightarrow d(y, y_n) < \frac{\varepsilon}{2}$. Hence

$d(y_m, y_{m+1}) \leq d(y, y_m) + d(y, y_{m+1}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Contradiction! — The source is the assumption in the first sentence in the proof, so that assumption is false. Hence the conclusion of the theorem is true. ∎

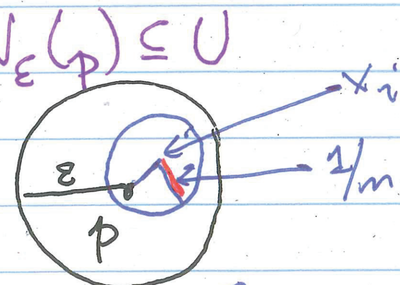
$\boxed{\text{Step 2 Under the hypothesis, the topology on } X \text{ has a countable base.}}$

For $m > 0$ let $\mathcal{U}_m$ be a finite set of nbhds. $N_{1/m}(x_i)$ which cover X, and let

countable family

$\mathcal{B} = \cup \, \mathcal{U}_m$. To see $\mathcal{B}$ is a base, let U be open in X, let $p \in U$, and let $\varepsilon > 0$ such that $N_\varepsilon(p) \subseteq U$. Choose $\frac{1}{m} < \frac{\varepsilon}{2}$ and x_i such that $p \in N_{1/m}(x_i)$.

CLAIM $N_{1/m}(x_i) \subseteq N_\varepsilon(p) \subseteq U$



Suppose $d(y, x_i) < 1/m$

Then $d(y,p) \leq \underbrace{d(y, x_i)}_{< 1/m} + \underbrace{d(x_i, p)}_{< 1/m} < \frac{2}{m} < 2 \cdot \left(\frac{\varepsilon}{2}\right) = \varepsilon$. ∎

Cor. to Step 2 (X, d) compact $\Longrightarrow$
there is a countable base for its topology

NOTE. A result known as the Urysohn metrization theorem implies a converse to Step 2:

If X is compact Hausdorff and the topology has a countable base, then X is homeomorphic to a (compact) metric space.

This follows by combining the discussion on page 195 of Munkres* following the definition with Theorems 32.3 and 34.1 in that book.

Not every compact Hausdorff space is homeomorphic to a metric space. The standard example with these properties is given as Example 105 in Steen and Seebach, Counterexamples in Topology.

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* Munkres, Topology (Second Edition).

Step 3. If $(X, \mathcal{T})$ has a countable base, then every open covering of X has a countable subcovering. (Lindelöf Property)

$\mathcal{U}$ = open covering of X, $\mathcal{B} = \{V_\beta\}$ a countable base. Let $\{W_j\} = \mathcal{B}_0 \subseteq \mathcal{B}$ be all subsets s.t. $W_j \subseteq U_\alpha$ for some U_α in $\mathcal{U}$. CLAIM $\{W_j\}$ is an open covering of X; this is true since $x \in X \Rightarrow x \in U_\alpha$ some α and some j so that $x \in W_j \subseteq U$. To get a countable subcovering, ~~for each W_j in $\mathcal{B}_0$ pick $j(\alpha)$ so that $W_j \subseteq U$~~ given W_j in $\mathcal{B}_0$, take $\alpha(j)$ so that $W_j \subseteq U_{\alpha(j)}$; then $\{U_{\alpha(j)}\}$ is a countable subcovering.

Conclusion of the proof.

Given an open covering $\mathcal{U}$ of X, take a countable subcovering $\mathcal{U}_0 = \{U_0, U_1, U_2, \dots\}$.

Suppose that $\mathcal{U}_0$ has no finite subcovering. Then each closed set $F_n = X - \bigcup_{i=0}^{n} U_i$ is infinite (otherwise there will be a finite subcovering).

Construct a sequence $\{a_n\}$ recursively so that $a_n \in F_n$ and $a_n \neq a_k$ for $k<n$. Then $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$ with some limit b. Since $\mathcal{U}_0$ covers X, $b \in U_M$ for some M. Pick $\varepsilon > 0$ so that $N_\varepsilon(b) \subseteq U_M$; then for some integer K, $k \geq K$ $\Longrightarrow d(b, a_{n_k}) < \varepsilon$ so that $a_{n_k} \in U_M$.

On the other hand, by construction $a_n \in F_{M+1}$ for $n \geq M+1$, which means $a_n \notin U_M$ for such n. If $n \geq K, M+1$ this yields a contradiction. The source of the contradiction is the assumption that $\mathcal{U}_0$ has no finite subcovering. Hence $\mathcal{U}_0$ (and also $\mathcal{U}$) must have a finite subcovering. ∎