SOLUTIONS TO EXERCISES FOR

MATHEMATICS 145A — Part 1

Fall 2014

2. Notations and terminology

Even-numbered exercises from Sutherland

2.2. *Discussion of the method of proof.* In most exercises asking for proofs that two sets A and B are equal, it is usually best to show that every element of A is also an element of B and vice versa.

Implementation of the method. Suppose that $x \in A$ and $x \notin V \cap A$. Since $x \notin X - A$ by hypothesis and $V \subset X$, this means that $x \notin (V \cap A) \cup (V \cap X - A) = V$, so that $x \in X - V$ and hence $x \in A \cap X - V$. Therefore $A - (V \cap A) \subset A \cap X - V$. — Now suppose that $x \in A \cap X - V$. Then $x \notin V$ and hence $x \notin A \cap V$, so that $x \in A - (V \cap A)$. Therefore $A \cap X - V \subset A - (V \cap A)$.■

2.4. We first show that the set on the left is contained in the set on the right. If $p \in U \times V$, then p is an ordered pair (u, v) with $u \in U$ and $v \in V$. Since $U \subset X$ and $V \subset Y$ it follows that $(u, v) \in X \times V$ and $(u, v) \in U \times Y$, so that p is contained in the intersection of these two sets. — Conversely, if $p = (x, y)$ is contained in this intersection, then $p \in U \times Y$ implies $x \in U$ and $p \in X \times V$ implies $y \in V$, and therefore $p = (x, y)$ is belongs to $U \times V$.■

2.6. Let W denote the union on the right hand side of the display for this exercise.

Proof that $U \cap V$ is contained in W. If $X \in U \cap V$ then $x \in U$ implies $x \in B_{j1}$ for some $j1$, and $x \in V$ implies $x \in B_{j2}$ for some $j2$, so that $x \in B_{j1} \cap B_{j2} \subset W$.

Proof that W is contained in $U \cap V$. If $x \in B_{j1} \cap B_{j2}$ for some $j1$ and $j2$ then $B_{j1} \subset U$ implies that $x \in U$ and $B_{j2} \subset V$ implies that $x \in V$; these combine to show that $x \in U \cap V$.■

2.8. Given a partition $\mathcal{P}$ of the set X, let $\mathcal{P}(x)$ be the subset in the partition which contains $x \in X$.

We shall begin by writing down the rules for passing between partitions and equivalence classes in clear symbolic terms: If $\sim$ is an equivalence relation on the set X, then the associated partition $\mathcal{P}\sim$ satisfies the condition $x \sim x'$ if and only if $\mathcal{P}\sim(x) = \mathcal{P}\sim(x')$, and if $\mathcal{P}$ is a partition with associated equivalence relation $\sim\mathcal{P}$, then $x \ \sim\mathcal{P}\ x'$ if and only if $\mathcal{P}(x) = \mathcal{P}(x')$.

We then have the chain of logical equivalences

$$x \sim x' \quad \Leftrightarrow \quad \mathcal{P}\sim(x) = \mathcal{P}\sim(x') \quad \Leftrightarrow \quad x \ \sim\mathcal{P}\sim\ x'$$

which shows that $\sim\mathcal{P}\sim \,=\, \sim$. Conversely, we also have

$$\mathcal{P}(x) = \mathcal{P}(x') \quad \Leftrightarrow \quad x \ \sim\mathcal{P}\ x' \quad \Leftrightarrow \quad \mathcal{P}\sim\mathcal{P}(x) = \mathcal{P}\sim\mathcal{P}(x')$$

which shows that $\mathcal{P}\sim\mathcal{P} = \mathcal{P}$.■

0. We shall first prove that $\mathcal{R}^{\#}$ is an equivalence relation.

The relation is reflexive. The definition of $\mathcal{R}^{\#}$ stipulates that $x\ \mathcal{R}^{\#}\ x$.

The relation is symmetric. By the preceding step we need only consider the case where $x \neq y$. If we are given a finite sequence $\{v_0\ \cdots\ , v_m\}$ as described in the definition such that $v_0 = x$ and $v_m = y$, then the reverse sequence with $w_j = v_{m-j}$ satisfies the criterion in the definition implies with $w_0 = y$ and $w_m = x$.

The relation is transitive. Suppose that we have a finite sequence $\{v_0\ \cdots\ , v_m\}$ as described in the definition such that $v_0 = x$ and $v_m = y$, and another finite sequence $\{u_0\ \cdots\ , u_n\}$ as described in the definition such that $u_0 = y$ and $u_n = z$, then we can concatenate (string together) the original sequences into a new sequence $\{t_0\ \cdots\ , t_{m+n}\}$ such that $t_j = v_j$ if $j \leq m$ and $t_j = u_{j-m}$ if $j \geq m$ (the formulas are consistent at the overlapping value m, for which $t_m = y$). This new sequence still satisfies the criterion in the definition with $t_0 = x$ and $t_{m+n} = z$.

To complete the proof, we need to verify that if $\mathcal{S}$ is an equivalence relation such that $x\ \mathcal{S}\ y$ whenever $x\ \mathcal{R}\ y$, then $u\ \mathcal{R}^{\#}\ v$ implies that $u\ \mathcal{S}\ v$. If $u = v$ then the latter follows because we are working with equivalence relations, which are reflexive. If $u \neq v$ and $u\ \mathcal{S}\ v$, let $\{t_0\ \cdots\ , t_m\}$ be a sequence starting with u and ending with v such that the for each i we have either $t_i\ \mathcal{R}\ t_{i+1}$ or $t_{i+1}\ \mathcal{R}\ t_i$. Then by the hypotheses on $\mathcal{S}$ we know that $t_i\ \mathcal{S}\ t_{i+1}$ for all i, and therefore by repeated application of the transitivity property we have $u\ \mathcal{S}\ v$.■

1. We shall refer to the file `solutions01w14.figures.pdf` for drawings which may help explain the underlying ideas; as usual, the proof must be written so that it does not formally depend upon these drawings.

The first step is to show that if $(i, j) \in \mathcal{E}$, then every point of the form $(i + t, j + t)$ in B — where t runs through all admissible integers such that the point in question belongs to B — also lies in $\mathcal{E}$. In other words, if $i' - j' = i - j$, then $(i', j')\ \mathcal{E}\ (i, j)$. For points in B the difference values $i - j$ are the 15 integers between ± 7, so this shows that there are at most 15 equivalence classes (in the first drawing, the squares with $i - j$ = CONSTANT are on the diagonal lines and have the same color). To prove the assertion in the first sentence, observe that $(i, j)\ \mathcal{R}\ (i + \varepsilon, j + \varepsilon)$ for $\varepsilon = \pm 1$ by definition, and by definition of $\mathcal{E}$ this yields $(i, j)\ \mathcal{E}\ (i + \varepsilon, j + \varepsilon)$. The statement for general values of t now follows by repeated application of the final assertion in the previous sentence and the transitivity of $\mathcal{E}$.

Next, let $\mathcal{F}$ be the binary relation with $(i', j')\ \mathcal{F}\ (i, j)$ if $i' + j'$ and $i + j$ are both even or both odd. This is an equivalence relation by one of the exercises in Sutherland and the fact that two ordered pairs are $\mathcal{F}$ are related if and only if they have the same values under the function $\varphi : B \to \{\text{EVEN}, \text{ODD}\}$ whose value is determined by whether $i + j$ is even or odd. The definition of $\mathcal{R}$ implies that if $(i, j)\ \mathcal{R}\ (p, q)$ then both $i+j$ and $p+q$ are even or odd, and therefore $(i, j)\ \mathcal{F}\ (p, q)$ whenever $(i, j)\ \mathcal{R}\ (p, q)$. By Exercise 0, it follows that $(i, j)\ \mathcal{F}\ (p, q)$ whenever $(i, j)\ \mathcal{E}\ (p, q)$, and since $\mathcal{F}$ has two equivalence classes the equivalence relation $\mathcal{E}$ must also have at least two equivalence classes.

Finally, we need to show that $\mathcal{E}$ has exactly two equivalence classes. The idea is similar to that of the first step; namely, if $(i, j) \in \mathcal{E}$, then every point of the form $(i + t, j - t)$ in B — where t runs through all admissible integers such that the point in question belongs to B — also lies in $\mathcal{E}$. The main difference in the argument is the need to observe that we also have $(i, j)\ \mathcal{R}\ (i + \varepsilon, j - \varepsilon)$

for $\varepsilon = \pm 1$ by the definition of $\mathcal{R}$. By the same reasoning as in the first step, this implies that if $i' + j' = i + j$, then $(i', j')\ \mathcal{E}\ (i, j)$. — To conclude the argument, it suffices to observe that the set of all $(i, j) \in B$ with $i + j = 9$ the difference $i - j$ takes all odd values between -7 and $+7$, while the set of all (i, j) with $i + j = 8$ takes all even values between -6 and $+6$ (in the second drawing, observe how the two lines with slope -1 cut through all the lines with slope $+1$). This proves that there are at most two equivalence classes for $\mathcal{E}$, and by the preceding paragraph there must be precisely two equivalence classes.■

2. It turns out that, in order to make things less repetitive, the best place to start is by observing that if $[x] = [y]$ then $x\ \mathcal{S}\ y$. This follows from the reflexive property of the equivalence relation $\mathcal{R}_2$. Note that this also yields the reflexive property for $\mathcal{S}$.

Suppose now that $x\ \mathcal{S}\ y$, so that $[x]\ \mathcal{R}_2\ [y]$. Since $\mathcal{R}_2$ is an equivalence relation, this means that $[y]\ \mathcal{R}_2\ [x]$, which in turn implies that $y\ \mathcal{S}\ x$. Finally, suppose that $x\ \mathcal{S}\ y$ and $y\ \mathcal{S}\ z$, so that $[x]\ \mathcal{R}_2\ [y]$ and $[y]\ \mathcal{R}_2\ [z]$. Since $\mathcal{R}_2$ is transitive we have $[x]\ \mathcal{R}_2\ [z]$, and this yields $x\ \mathcal{S}\ z$, so that $\mathcal{S}$ is an equivalence relation on X.■

3. More on sets and functions

Even-numbered exercises from Sutherland

3.2. *Discussion of the method of proof.* Usually the two best approaches are to work with algebraic expressions (for this problem and the next one, this involves manipulating equations and inequalities) or to start by drawing pictures reflecting the conditions in (individual parts of) the exercise.

Implementation of the method. The sets in the exercise are given respectively as follows:

$$[0,1] \qquad [-1,1]\ , \qquad \cup_k\ [2k\pi, (2k+1)\pi]\ ,$$

$$\cup_k\ \left(\left[2k\pi, 2k + (\tfrac{1}{6})\pi\right] \cup \left[2k - \tfrac{5}{6})\pi, (2k+1)\pi\right]\right)\ , \qquad \mathbb{R}$$

One effective way of finding these answers is to draw a graph of the sine function; in the next to last part it is also necessary to realize that between 0° and 180° (more formally, between 0 and 2π radians) the values of the sine function lie in $[0, \frac{1}{2}]$ precisely for angles less than 30° ($= \frac{\pi}{6}$ radians) or greater than 150° ($= \frac{5\pi}{6}$ radians), while for angles strictly between 180° and 360° (*i.e.*, between π and 2π radians) the sine function is negative.■

3.4. The sets in the exercise are given respectively as follows:

$$[0,2]\ , \qquad [-1,1]\ , \qquad \left[-\sqrt{5}/5,\ \sqrt{5}/5\right]$$

Once again, the most effective way to determine the answers is to draw a graph and mark off the limiting values the sets $[0,1]$, $[0,1] \times [0,1]$ and U. In particular, the third set is the portion of the graph of $y = 2x$ contained in the disk with equation $x^2 + y^2 \leq 1$. The x-coordinates of the points where the graph meets the boundary circle satisfy the equation $5x^2 = 1$, and this leads one to the (informal) conclusion that the x-coordinates of points in the inverse image lie between $\pm\sqrt{5}/5$.■

Note. In the preceding exercises we have only figured out what the relevant sets are, and we have not given rigorous proofs. We have not done so because the problems only ask for *descriptions* of the images or inverse images. For each example the detailed verifications are routine and somewhat tedious. In particular, for the second part of **3.2** one needs to say that the sine function takes every value between -1 and $+1$ as x ranges over all nonnegative real numbers, for the second part of **3.4** one needs to look at all points of the form $(x, 2x)$ such that $0 \leq x \leq 1$ and $0 \leq 2x \leq 1$, and for the third part of **3.4** one needs to look at all points $(x, 2x)$ such that $5x^2 = x^2 + (2x)^2 \leq 1$. — The reader should try to fill in these details on his or her own.

3.6. We first show that if f is injective then $f^{-1}[\,f[A]\,] = A$. ($\supset$) If $a \in A$ then $b = f(a) \in f[A]$; clearly $f(a) \in f[A]$, and therefore a lies in the inverse image of $f[A]$. ($\subset$) Suppose that $x \in f^{-1}[\,f[A]\,]$. Then $f(x) \in f[A]$, so that $f(x) = f(a)$ for some $a \in A$. Since f is injective this implies that $x = a$, which means that $x \in A$.

Conversely, we shall show that if $f^{-1}[\,f[A]\,] = A$ for all A then f is injective. Actually, we shall prove the contrapositive statement. Suppose that f is NOT injective, and let $x, y \in X$ be distinct points such that $f(x) = f(y)$. In this case we have

$$\{x\} \quad \neq \quad \{x, y\} \quad \subset \quad f^{-1}[\,f[\{x\}]\,]$$

so there is some subset $A \subset x$ for which $f^{-1}[\,f[A]\,] \neq A$.∎

3.8. Suppose first that $f[A - B] = f[A] - f[B]$. Then we have

$$f[A - B] \cap f[B] \quad = \quad (f[A] - f[B]) \cap f[B] \quad = \quad \emptyset$$

which yields the "only if" implication.

Now suppose that $f[A - B] \cap f[B] = \emptyset$. Then $f[A - B] \subset f[A] - f[B]$ follows immediately. On the other hand, if $y \in f[A] - f[B]$, write $y = f(a)$ for $a \in A$. Then the condition in the first sentence of the paragraph implies that y cannot have the form $f(b)$ for any $b \in B$, which means that the point a must lie in $A - B$, and therefore we have $y \in f[A - B]$.∎

3.10. To streamline notation, for each $y \in Y$ we shall denote the level set $f^{-1}[\{y\}]$ by $L(y)$. Since f is onto, each set $L(y)$ is nonempty, and since the values of f lie in Y we know that the union of the sets $L(y)$ is equal to X. It remains to show that two level sets are disjoint or identical. Suppose that $L(y_1)$ and $L(y_2)$ are not disjoint, and let x be a common point. Then we have $f(x) = y_1$ and $f(x) = y_2$, which combine to imply that $y_1 = y_2$ and hence $L(y_1) = L(y_2)$.∎

Additional exercise(s)

1. The empty set is an initial object because for each set S there is a unique function $\emptyset \to S$; namely, the function whose graph is the empty set. A nonempty set A cannot be an initial object, for in this case there are always two maps into $\{1, 2\}$; namely the constant functions whose values everywhere are 1 and 2 respectively.

A one point set $\{p\}$ is a terminal object, for if A is a nonempty set then the only map into $\{p\}$ is the map whose value is always p, and if A is empty then there is only one map by the preceding paragraph. On the other hand, if a set B contains more than one element, then for every nonempty set A there are at least two functions by the reasoning in the previous paragraph, so B cannot be a terminal object.∎

2. (i) Following the hint, define f by $f(a, 1, c) = (a, c, 1)$ if $a \in A$ and $f(b, 2, c) = (b, c, 2)$ if $b \in B$. We can show this map is a 1–1 correspondence by constructing the inverse function g, which is given by $g(a,c,1) = (a,1,c)$ if $a \in A$ and $g(b,c,2) = (b,2,c)$ if $b \in B$. Checking that $g \circ f$ and $f \circ g$ are identity mappings is straightforward.■

(ii) The defining formulas show that there is at most one such function because they give the values at all points of $A \amalg B$, but we also need to verify that we actually have a function, and to do so we need to describe its graph. Let Γ_f and Γ_g denote the graphs of f and g respectively, and consider the image G of

$$\Gamma_f \amalg \Gamma_g \quad \subset \quad (A \amalg C) \times (B \amalg C)$$

under the mapping h described in the first part of the problem. To see that this is the graph of a function, it is only necessary to check that for each point p of $A \amalg B$ there is a uniqe point in G whose first coordinate is p. If p comes from A, then the only such point in G is $(a, 1, f(a))$, and if p comes from B, then the only such point is $(b, 2, g(b))$.■

4. Review of some real analysis

Even-numbered exercises from Sutherland

Solutions are not given for exercises **4.6**, **4.10**, **4.16** and **4.18**; the reasons for exclusion are that the exercises in question seem more appropriate for advanced undergraduate courses like Mathematics 151A or 171. The same might be said for **4.12** and **4.14**, but in these cases it seemed wortwhile to present a few examples involving functions which are **not** continuous.

4.2. Without loss of generality, we might as well assume $\sup A \geq \sup B$, for if the reverse inequality is true we can obtain the same conclusion by reversing the roles of A and B in the argument.

To see that $\sup A \leq \sup A \cup B$, note that more generally $X \subset Y$ implies $\sup X \leq \sup Y$, for if K is an upper bound for Y, then K is an upper bound for X. Hence $\sup A \cup B$ is an upper bound for A, which means that $\sup A \cup B \geq \sup A$.

Conversely, we need to show that $\sup A \geq \sup B$ implies $\sup A \cup B \leq \sup A$. Suppose that K is an upper bound for A; then the condition in the previous sentence implies that $K \geq \sup B$, so that K is also an upper bound for B and hence K is an upper bound for $A \cup B$. In particular, this implies that $\sup A \geq \sup B$.■

4.4. We shall denote the four examples symbolically as (n, X), where n is the row in which the example appears and $X = L$ or R depending upon whether the example is on the left or right respectively.

EXAMPLE $(1, L)$. The defining inequality is $0 \geq x^2 - 2x + 1 = (x-1)^2$, and the only real number satisfying this inequality is $x = 1$, so 1 is the least upper bound.■

EXAMPLE $(1, R)$. The defining inequality is $x^2 + 2x - 1 \leq 0$ or equivalently $(x+1)^2 \leq 2$, so the least upper bound is the value of x for which $x + 1 = \sqrt{2}$. This means that the least upper bound is $\sqrt{2} - 1$.■

EXAMPLE $(2, L)$. The least upper bound is the value of x such that $x^3 = 8$, which means that $x = 2$.■

EXAMPLE $(2, R)$. There is no least upper bound, (unless we pass to the extended real number system $\mathbb{R} \cup \{-\infty, +\infty\}$, in which case the least upper bound is $+\infty$). This amounts to saying that there are arbitrarily large values of x such that $x \sin x < 1$. In fact there are arbitrarily large values of x such that $x \sin x = 0$; specifically, this is true for $x = n\pi$, where n is an arbitrary positive integer.■

Note. As nonsensical as it may seem, if we view the least upper bound of a set of real numbers as an element of the extended real number system, then the least upper bound of the empty set in the extended real number is $-\infty$, for the statement "$-\infty$ is an upper bound for the empty set" is vacuously true (since there are no real numbers in the empty set, it is formally correct to say that $-\infty$ is greater than every real number in the empty set). For the same reasons, $-\infty$ is also the greatest lower bound of the empty set.

4.6. Omitted (see the comments at the top of this section).

4.8. *Discussion of the method of proof.* By Corollary 4.7 in Sutherland we know that if $x < y$ then there is a rational number a such that $x < a < y$. We need to find an irrational number which also lies in this range. Since $\sqrt{2}$ is one of the simplest examples of an irrational number, one promising approach is to modify a by some arithmetic construction involving $\sqrt{2}$.

Implementation of the method. Start off by observing that by Corollary 4.7 we can find a second rational number b such that $a < b < y$. Next, observe that since the square root function is an increasing function on the set of nonnegative real numbers, we have $1 = \sqrt{1} < \sqrt{2} < 2$; it follows that $0 < \sqrt{2} - 1 < 1$. If

$$c \; = \; a \; + \; (\sqrt{2} - 1) \cdot (b - a)$$

then $a < c < c + (b - a) = b$, so it only remains to show that c is irrational. — Assume that c is rational. Then it follows that $c - a = (\sqrt{2} - 1) \cdot (b - a)$ is rational, which implies that

$$\sqrt{2} \; = \; 1 \; + \; \frac{c - a}{b - a}$$

is also rational. This yields a contradiction because $\sqrt{2}$ is irrational; the source of the contradiction is the assumption that c is a rational number, and thus we see that c must be irrational.■

4.10. Omitted (see the comments at the top of this section).

4.12. *Discussion of the method of proof.* At first the statement of this exercise may appear to contradict Proposition 4.33, so it will be helpful to see why there is no contradiction. The proposition states that if f is continuous at a and g is continuous at $b = f(a)$, then the composite $g \circ f$ is continuous at a. Symbolically this can be written in the form

$$\lim_{x \to a} f(x) \; = \; f(a) \; = \; b \; , \lim_{y \to b} g(y) \; = \; g(b) \; = \; c \qquad \Longrightarrow \qquad \lim_{x \to a} g \circ f(x) \; = \; g \circ f(a) \; = \; c \; .$$

Therefore at least one of the functions f, g must be discontinuous, and since each function has decent limit behavior an example is likely to involve at least one function which is obtained by taking a continuous function and changing its value at one point to make it discontinous (for example, this can be done by adding 1 to the value of the function at the point to manufacture a discontinuity); since the value of the limit only depends upon the behavior of the function at points close, but not equal, to the point being approached, this change will not affect the limits of the function anywhere. In cases like this which involve an exotic example, it is often effective to look through the text for potentially useful examples; one candidate is Example 4.22, which looks like $g(y) = y$ except when $y = 0$, in which case we let $g(0) = 1$. Next, we might want to choose f so that $g \circ f$ is not continuous at a. In order for this to happen, we need a function f which is not constant but satisfies $f(x_n) = 0$ for some sequence of points $\{x_n\}$ whose limit is zero. The function $f(x) = x \sin(1/x)$ in Example 4.29 is one potential candidate.

Implementation of the method. As in the preceding discussion, choose g and f to be the functions in Examples 4.22 and 4.24 respectively, so that both have limit equal to 0 as $x \to 0$. In other words, we have $a = b = c = 0$, but the limit of $g \circ f$ as $x \to 0$ is not defined. This is true because

$$g \circ f \left(\frac{2}{\pi\,(4n+1)} \right) \quad = \quad \frac{2}{\pi\,(4n+1)} \quad \longrightarrow \quad 0 \qquad \text{as} \qquad n \to \infty$$

but on the other hand

$$g \circ f \left(\frac{1}{n\,\pi} \right) \quad = \quad 1 \qquad \text{for all} \qquad n.$$

By Lemma 4.25, if a function $h(t)$ has a limit L as $t \to a$, then for each sequence $\{t_n\}$ such that $t_n \to a$ but $t_n \neq a$ (for **all** n) we have $h(t_n) \to L$. By the preceding discussion this does not happen for the function $g \circ f$ when $a = 0$, and therefore $g \circ f$ does not have a limiting value of zero (or indeed any limiting value) as $x \to 0$.■

4.14. *Discussion of the method of proof.* It will be helpful to have the following **Squeeze Principle** for evaluating limits of functions; there is a similar principle for limits of sequences, and the statement and proof are left to the reader as an exercise in adapting a proof. The Squeeze Principle is often extremely useful for evaluating limits without getting into messy estimates for the values of functions.

Suppose that we have three functions defined on the open interval $(a-c, a+c)$ except possibly at a and that they satisfy $f(x) \leq g(x) \leq h(x)$ when $0 < |x-a| < c$. If $\lim_{x\to a}\ f(x) = L = \lim_{x\to a}\ h(x)$, then $\lim_{x\to a}\ g(x) = L$.

Proof of the Squeeze Principle. By the definition of limit, for each $\varepsilon > 0$ there exist $\delta_f, \delta_h > 0$ such that $0 < |x-a| < \delta_f$ implies $|f(x) - L| < \varepsilon$, and furthermore $0 < |x-a| < \delta_h$ implies $|h(x) - L| < \varepsilon$. If δ is the smaller of δ_f and δ_h, then $0 < |x-a| < \delta$ implies

$$L - \varepsilon \quad < \quad f(x) \quad \leq \quad h(x) \quad \leq \quad g(x) \quad < \quad L + \varepsilon$$

which means that $|g(x) - L| < \varepsilon$, so that the limit of g is also equal to L.■

Implementation of the method. In the first case we have $-x \leq f(x) \leq x$ because the sine function takes values in the interval $[-1, 1]$. There is no question about the continuity of the function at $x = 0$, and at $x = 0$ we can apply the squeeze principle for the inequality chain $-x \leq f(x) \leq x$.

For the second function we shall use Lemma 4.25; the idea is to construct a sequence of points $t_n \to 0$ such that $g(t_n) = 1$ for all n. All we have to do is take $t_n = 1/2n\pi$ where n runs through the positive integers.■

4.16. Omitted (see the comments at the top of this section).

4.18. Omitted (see the comments at the top of this section).

Additional exercise(s)

1. Since $n(k)$ is strictly increasing we must have $n(k) \geq k$ for all k (Proof by induction: $n(0) \geq 0$, and if $n(m) \geq n$ then $n(m+1) >\geq n(m) + 1 \geq m + 1$ because $n(k)$ is strictly increasing). By the definition of a limit for a sequence, for every $\varepsilon > 0$ there is a positive integer N such that $n \geq N$ implies $|a_n - L| < \varepsilon$. Therefore if $k \geq N$ then $n(k) \geq n$, so that $|a_{n(k)} - L| < \varepsilon$.■

NOTE. The contrapositive forms of this result are particularly useful for showing that a specific sequence has no limit. If either (a) there is a subsequence with no limit, or (b) there are two subsequences with different limits, then the original sequence cannot have a limit.■

2. (i) The sequence of left hand endpoints $\{a_n\}$ is bounded from above by b_k, where k is an arbitrary nonnegative integer, while the sequence of right hand endpoints $\{b_n\}$ is bounded from below by a_k, where k is an arbitrary nonnegative integer. Therefore the set of left hand endpoints has a least upper bound A, and $A \leq b_k$ for all k. Similarly, the set of right hand endpoints has a greatest lower bound B, and by the preceding sentence we must have $A \leq B$. Therefore every point p satisfying $A \leq p \leq B$ must lie in each of the intervals. — Note that $A = B$ is possible; for example, consider the intervals $\left[2^{-k}, 2^k\right]$.

(ii) The answer is definitely **NO**. We shall give a counterexample involving $\sqrt{2}$. Since there is always a rational number between two real numbers, for each n we can find rational number a_n and b_n such that

$$\sqrt{2} \; - \; \left(\tfrac{1}{2}\right)^n \quad < \quad a_n \quad < \quad \sqrt{2} \quad < \quad b_n \quad < \quad \sqrt{2} \; + \; \left(\tfrac{1}{2}\right)^n \; .$$

It follows that if $p \in [a_n, b_n]$ then $|\sqrt{2} - p| \leq \left(\frac{1}{2}\right)^n$, so if p lies in each interval then this inequality holds for every n. But the latter implies that $|p - \sqrt{2}| = 0$, so that $p = \sqrt{2}$ and the latter is the only point which lies on each interval. Since $\sqrt{2}$ is irrational, there is no rational number which lies on all of the intervals.■