SOLUTIONS TO EXERCISES FOR

MATHEMATICS 145A — Part 5

Winter 2014

11. The Hausdorff condition

Exercises from Sutherland

`See the next three pages.`

Solutions to Chapter 11 exercises

11.1 Suppose that x, y are distinct points in a space X with the indiscrete topology. Then there are no disjoint open sets U, V with $x \in U$ and $y \in V$, since the only open set containing x is X, which also contains y.

11.2 (a) Suppose that X is Hausdorff and $\{x\}$ is a singleton set in X. For any $y \neq x$ in X, there exist disjoint open sets U_y, V_y such that $x \in U_y$, $y \in V_y$. In particular $x \notin V_y$. Now $X \setminus \{x\}$ is the union, over all $y \in X \setminus \{x\}$ of the V_y and hence is open so $\{x\}$ is closed.

(b) Now if X is finite and Hausdorff then any subset of X is a finite union of singletons, hence by (a) a finite union of closed sets, hence is closed in X. Then by taking complements we see that every subset of X is open, so X has the discrete topology.

11.3 We can prove this by induction on n. When $n = 2$ the conclusion is simply the Hausdorff condition. Suppose the result true for a given integer $n \geqslant 2$, and let $x_1, x_2, \ldots, x_{n+1}$ be distinct points in X. By inductive hypothesis, for $i = 1, 2, \ldots, n$ there exist pairwise disjoint open sets W_i with $x_i \in W_i$. Also, by the Hausdorff condition, for each $i = 1, 2, \ldots, n$ there exist disjoint open sets S_i, T_i such that $x_i \in S_i$, $x_{n+1} \in T_i$. For each $i = 1, 2, \ldots, n$ put $U_i = S_i \cap W_i$, and put $U_{n+1} = \bigcap_{i=1}^{n} T_i$. Then $U_1, U_2, \ldots, U_n$ are pairwise disjoint since $W_1, W_2, \ldots, W_n$ are, and for each $i = 1, 2, \ldots, n$ we have $U_i \cap U_{n+1} = \emptyset$ since $U_i \subseteq S_i$ and $U_{n+1} \subseteq T_i$. Also, by construction each of $U_1, U_2, \ldots, U_{n+1}$ is open in X. Thus $U_1, U_2, \ldots, U_{n+1}$ are pairwise disjoint. This completes the inductive step.

11.4 (a) Suppose that X is Hausdorff and A is a subspace of X. For any distinct points $a_1, a_2 \in A$ there are disjoint open subsets U, V of X with $a_1 \in U$, $a_2 \in V$. Now by definition of the subspace topology $A \cap U$, $A \cap V$ are open in A; also, $a_1 \in A \cap U$ and $a_2 \in A \cap V$, and $A \cap U$, $A \cap V$ are disjoint since U, V are. Hence A is Hausdorff.

(b) First suppose that X and Y are Hausdorff spaces, and let (x_1, y_1), (x_2, y_2) be distinct points in the topological product $X \times Y$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$ (or both). If $x_1 \neq x_2$ then there exist disjoint open subsets U, V of X with $x_1 \in U$, $x_2 \in V$. Then $U \times Y$, $V \times Y$ are disjoint open subsets of $X \times Y$ with $(x_1, y_1) \in U \times Y$ and $(x_2, y_2) \in V \times Y$. There is an entirely similar argument if $y_1 \neq y_2$.

Conversely suppose that $X \times Y$ is Hausdorff. Let x_1, x_2 be distinct points in X. Then since Y is non-empty, there is some $y \in Y$ with $(x_1, y) \neq (x_2, y)$. Since $X \times Y$ is Hausdorff, there exists disjoint open subsets W_1, W_2 of $X \times Y$ such that $(x_1, y) \in W_1$, $(x_2, y) \in W_2$. Since W_1 is open in $X \times Y$ and contains (x_1, y) there exist open subsets U_1, V_1 of X, Y respectively such that $(x_1, y) \in U_1 \times V_1 \subseteq W_1$. Similarly there exist open subsets U_2, V_2 of X, Y such that $(x_2, y) \in U_2 \times V_2 \subseteq W_2$. Now $x_1 \in U_1$, $x_2 \in U_2$, and $U_1 \cap U_2 = \emptyset$ (since if $x \in U_1 \cap U_2$ we would have $(x, y) \in W_1 \cap W_2$). This proves that X is Hausdorff. Similarly Y is Hausdorff.

(c) Let x_1, x_2 be distinct points in X. Since f is injective, $f(x_1)$ and $f(x_2)$ are distinct points in Y. Since Y is Hausdorff, there are disjoint open subsets V_1, V_2 of Y with $f(x_1) \in V_1$, $f(x_2) \in V_2$. By continuity of f the sets $U_1 = f^{-1}(V_1)$ and $U_2 = f^{-1}(V_2)$ are open in X. Moreover, $x_1 \in U_1$ since $f(x_1) \in V_1$, and similarly $x_2 \in U_2$. Finally, U_1 and U_2 are disjoint, since f is injective and V_1, V_2 are disjoint.

(d) This follows from (c), since if $f : X \to Y$ is a homeomorphism then both f and $f^{-1} : Y \to X$ are continuous injective maps, so if Y is Hausdorff so is X and vice-versa.

11.5 We prove that $X \times Y \setminus G_f$ is open in $X \times Y$. (Once we have opted to try this, the rest of the proof 'follows its nose'.) So let $(x, y) \in X \times Y \setminus G_f$. It is enough, by Proposition 7.2, to show that for every point $(x, y) \in X \times Y \setminus G_f$ there exists an open subset W of $X \times Y$ with $(x, y) \in W \subseteq X \times Y \setminus G_f$. So let $(x, y) \in X \times Y \setminus G_f$. Then $(x, y) \notin G_f$, so $f(x) \neq y$. Since Y is Hausdorff, there exist disjoint open subsets V_1, V of Y such that $f(x) \in V_1$, $y \in V$. Since f is continuous, $U = f^{-1}(V_1)$ is open in X. Note that $x \in U$ since $f(x) \in V_1$. Also, $y \in V$. Write $W = U \times V$. Then $(x, y) \in U \times V = W$, and $W \subseteq X \times Y \setminus G_f$ since if $(x', y') \in W$ then $x' \in U$ and $y' \in V$ so $f(x') \in V_1$ and $y' \in V$, but $V_1 \cap V = \emptyset$ so $f(x') \neq y'$, which says that $(x', y') \notin G_f$.

11.6 (a) Clearly x is in the intersection S of all open subsets of X containing x. Suppose that $y \neq x$. Then by the Hausdorff condition there are disjoint open sets U, V such that $x \in U$ and $y \in V$. In particular there is an open subset U containing x which does not contain y, so $y \notin S$. This shows that $S = \{x\}$.

(b) Let X be an infinite set and consider it as a space with the co-finite topology. Then X is not Hausdorff, by Example 11.6. But for any $x \in X$ let S be the intersection of all open subsets of X containing x. Then $x \in S$, but if $y \neq x$ then $X \setminus \{y\}$ is a set with finite complement (namely $\{y\}$) hence is open in X and contains x but not y, so $y \notin S$. Hence $S = \{x\}$.

11.7 (a) Suppose first that X is a Hausdorff space. We shall prove that $X \times X \setminus \Delta$ is open in $X \times X$, from which it follows that Δ is closed in $X \times X$. So let $(x, y) \in X \times X \setminus \Delta$. By Proposition 7.2 it is enough to show that there is an open set W of $X \times X$ with $(x, y) \in W \subseteq X \times X \setminus \Delta$. Now $(x, y) \notin \Delta$ so $y \neq x$. Since X is Hausdorff there exist disjoint open subsets U, V of X such that $x \in U$, $y \in V$. Then $W = U \times V$ is an open set in $X \times X$, and $(x, y) \in U \times V$. Moreover $W \subseteq X \times X \setminus \Delta$ since if $(x', y') \in W$ then $x' \in U$, $y' \in V$ and $U \cap V = \emptyset$, so $y' \neq x'$, which says $(x', y') \notin \Delta$.

Conversely suppose that Δ is closed in $X \times X$. Then $X \times X \setminus \Delta$ is open in $X \times X$. Now let x, y be distinct points of X. Then $y \neq x$ so $(x, y) \notin \Delta$. Hence (x, y) is in the open set $X \times X \setminus \Delta$ and by definition of the product topology there exist open subsets U, V of X such that $(x, y) \in U \times V \subseteq X \times X \setminus \Delta$. Now $x \in U$, $y \in V$, and $U \cap V = \emptyset$ since if $z \in U \cap V$ then $(z, z) \in \Delta \cap (U \times V)$ - but this set is empty. So X is Hausdorff.

(b) Consider the characteristic function $\chi_A : X \times X \to \mathbb{S}$ of the set $A = X \times X \setminus \Delta$. The open sets in $\mathbb{S}$ are $\emptyset$, $\{1\}$, $\mathbb{S}$. Now $\chi_A^{-1}(\emptyset) = \emptyset$, $\chi_A^{-1}(1) = A$ and $\chi_A^{-1}(\mathbb{S}) = X \times X$. Thus χ_A is continuous iff $X \times X \setminus \Delta$ is open in $X \times X$, that is iff Δ is closed in $X \times X$, and by (a) this holds iff X is Hausdorff.

11.8 Suppose for a contradiction that $f(x) \neq g(x)$ for some $x \in \overline{A}$. Since Y is Hausdorff, there exist disjoint open subsets U, V of Y such that $f(x) \in U$, $g(x) \in V$. By continuity of f the sets $f^{-1}(U)$ and $g^{-1}(V)$ are open in X; also, $x \in f^{-1}(U)$ since $f(x) \in U$, and similarly $x \in g^{-1}(V)$. So $W = f^{-1}(U) \cap g^{-1}(V)$ is open in X and $x \in W$. Now $x \in \overline{A}$, so there is a point a in $A \cap W$. But now since $a \in W \subseteq f^{-1}(U)$ we have $f(a) \in U$ and similarly $g(a) \in V$. This contradicts $U \cap V = \emptyset$ since $f(a) = g(a)$ for all $a \in A$. Hence $f(x) = g(x)$ for all $x \in \overline{A}$.

11.9 Since f_A and f_B are continuous, so is g. So since $(0, \infty)$ and $(-\infty, 0)$ are open in $\mathbb{R}$ we know that $g^{-1}(0, \infty)$ and $g^{-1}(-\infty, 0)$ are open in X. Also, A, B are closed sets, so $\overline{A} = A$ and $\overline{B} = B$. Now from Exercise 6.16, $f_A(x) \geqslant 0$ for all $x \in X$ and $f_A(x) = 0$ iff $x \in A$. Similarly $f_B(x) \geqslant 0$ for all $x \in X$ and $f_B(x) = 0$ iff $x \in B$. Hence, since A, B are disjoint, for $x \in A$ we have $f_A(x) = 0$ and $f_B(x) > 0$, so $g(x) < 0$. Similarly for $x \in B$ we have $g(x) > 0$. Thus $A \subseteq g^{-1}(-\infty, 0)$ and $B \subseteq g^{-1}(0, \infty)$. Finally, $g^{-1}(-\infty, 0)$ and $g^{-1}(0, \infty)$ are clearly disjoint (if $x \in g^{-1}(-\infty, 0)$ then $g(x) < 0$ and if $x \in g^{-1}(0, \infty)$ then $g(x) > 0$).

11.10 This follows from previous results. First, the composition $(f \times g) \circ \Delta : X \to Y \times Y$ is continuous, where $\Delta : X \to X \times X$ is the diagonal map of X. (This uses Propositions 10.13, 10.12 and 8.4). Also, the diagonal set Δ_Y in $Y \times Y$ is closed in $Y \times Y$ by Exercise 11.7(a). But $C = \Delta^{-1}(f \times g)^{-1}(\Delta_Y)$ since $x \in C$ iff $f(x) = g(x)$ iff $(f \times g)(x, x) \in \Delta_Y$ iff $((f \times g) \circ \Delta)(x) \in \Delta_Y$ iff $x \in \Delta^{-1}(f \times g)^{-1}(\Delta_Y)$. So C is closed in X, by Proposition 9.5. In particular we can apply this for any continuous self-map $f : X \to X$, taking $g : X \to X$ to be the identity function; in this case $C = \{x \in X : f(x) = x\} = F$, and we get that the fixed-point set F is closed in X.

Additional exercise(s)

1. Following the hint, let $f(x) = 1/x$ for $x \neq 0$ and set $f(x) = 0$. Then the graph is the union of the closed sets $\{(x, y) \mid xy = 1\}$ and $\{(0, 0)\}$, so the graph is closed. However, the function is not continuous at 0, for if f were continuous at 0 then there would be some $\delta > 0$ such that $|f(x)| < \frac{1}{2}$ for $|x| < \delta'$ for all δ' such that $0 < \delta' \leq \delta$. Since $|f(x)| > \frac{1}{2}$ for $0 < |x| < 2$, the inequality $|f(x)| < \frac{1}{2}$ is never satisfied if we choose $\delta' < 2$ and $|x| < \delta'$.■

2. If X is Hausdorff and $x \neq y$, then there are disjoint open neighborhoods U and V of x and y respectively. Since $\mathcal{B}$ is a base for the topology, there are open sets U' and V' in $\mathcal{B}$ such that $x \in U' \subset U$ and $y \in V' \subset V$. This proves the "only if" implication. — The reverse implication is trivial because basic open subsets are open.■

3. Since A and B are closed subsets of X and Y respectively, by continuity the inverse images $\pi_X^{-1}[A]$ and $\pi_Y^{-1}[B]$, are closed in $X \times Y$. As indicated in the hint, $A \times B$ is the intersection of $\pi_X^{-1}[A]$ and $\pi_Y^{-1}[B]$, and therefore the set $A \times B$ is closed in $X \times Y$.■

12. Connected spaces

Exercises from Sutherland

`See the next six pages.`

Solutions to Chapter 12 exercises

12.1 (i) has the partition $\{B_1((1,\,0)),\ B_1((-1,\,0))\}$ so it is not connected, and hence not path-connected. The others are all path-connected and hence also connected. We can see this for (ii) and (iii) by observing that any point in $\overline{B_1((1,\,0))}$ may be connected by a straight-line segment to the centre $(1,\,0)$ (and this segment lies entirely in $\overline{B_1((1,\,0))}$), any point in $X = \overline{B_1((-1,\,0))}$ or $B_1((-1,\,0)))$ can be connected by a straight-line segment in X to the centre $(-1,\,0)$ (and this segment lies entirely in $\overline{B_1((-1,\,0))}$ or in $B_1((-1,\,0)))$. Moreover the points $(-1,\,0)$ and $(1,\,0)$ can be connected by a straight-line segment which lies entirely in $\overline{B_1((-1,\,0))} \cup B_1((1,\,0))$, hence certainly in $\overline{B_1((-1,\,0))} \cup \overline{B_1((1,\,0))}$.

To see that (iv) is path-connected, note that any point $(q,\,y) \in \mathbb{Q} \times [0,\,1]$ can be connected by a straight-line segment within $\mathbb{Q} \times [0,\,1]$ to the point $(q,\,1)$, and that any two points in the line $\mathbb{R} \times \{1\}$ can be connected by a straight-line segment entirely within this line.

Finally, to see that the set S in (v) is path-connected, it is enough to show that any point $(x,\,y) \in S$ can be connected to the origin $(0,\,0)$ by a path in S. If $x \in \mathbb{Q}$ we first connect $(x,\,y)$ to $(x,\,0)$ by a (vertical) straight-line segment in S, then $(x,\,0)$ to $(0,\,0)$ by a (horizontal) straight-line segment in S. If $x \notin \mathbb{Q}$ then $y \in \mathbb{Q}$ and we first connect $(x,\,y)$ to $(0,\,y)$ by a (horizontal) straight-line segment in S, then $(0,\,y)$ to $(0,\,0)$ by a (vertical) straight-line segment in S.

12.2 This follows from either Exercise 10.7 (a) or 10.7 (b), since each of A, B is open (and also closed) in X.

12.3 Suppose that X is an infinite set with the co-finite topology, and let U, V be non-empty open subsets of X. Then $X \setminus U$ and $X \setminus V$ are both finite. So $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$ is finite. This shows that $U \cap V$ is non-empty (indeed, infinite). Hence there is no partition of X, and this says X is connected.

12.4 Suppose that $\{U,\,V\}$ is a partition of $(X,\,\mathcal{T}_1)$. Since $\mathcal{T}_1 \subseteq \mathcal{T}_2$ it follows that $U,\,V \in \mathcal{T}_2$, so $\{U,\,V\}$ is a partition of $(X,\,\mathcal{T}_2)$ also. Hence if $(X,\,\mathcal{T}_2)$ is connected then so is $(X,\,\mathcal{T}_1)$.

The converse is not true in general: for a counterexample, consider $X = \{0,\,1\}$ with $\mathcal{T}_1$ the indiscrete topology and $\mathcal{T}_2$ the discrete topology. Then $\mathcal{T}_1 \subseteq \mathcal{T}_2$ but $(X,\,\mathcal{T}_1)$ is connected while $(X,\,\mathcal{T}_2)$ is not connected.

12.5 We may prove that this holds for any finite integer n by induction: the result is certainly true when $n = 1$ and if it holds for a given n then for $n+1$ it follows from Proposition 12.16 applied to the connected sets $A_1 \cup A_2 \cup \ldots \cup A_n$ (which is connected by inductive hypothesis) and A_{n+1}; these have a non-empty intersection since $A_n \cap A_{n+1} \neq \emptyset$.

The analogous result is also true for an infinite sequence (A_i) of connected sets such that $A_i \cap A_{i+1} \neq \emptyset$ for every positive integer i. For suppose $\{U, V\}$ is a partition of $\bigcup_{i=1}^{\infty} A_i$. For each $i \in \mathbb{N}$ we have either $A_i \subseteq U$ or $A_i \subseteq V$, since otherwise $\{A_i \cap U, A_i \cap V\}$ would be a partition of A_i. Let I_U be the subset of $i \in \mathbb{N}$ such that $A_i \subseteq U$ and let I_V be the analogous set for V. Suppose without loss of generality that $1 \in I_U$ (otherwise exchange the names of U, V). Now $n \in I_U$ implies $n+1 \in I_U$, for if $A_n \subseteq U$ then the connected set $A_n \cup A_{n+1}$ is also contained in U, since otherwise $\{(A_n \cup A_{n+1}) \cap U, (A_n \cup A_{n+1}) \cap V\}$ would partition $A_n \cup A_{n+1}$. It follows that $\mathbb{N} \subseteq I_U$, so $\bigcup_{i=1}^{\infty} A_i \subseteq U$, contradicting the assumption that $\{U, V\}$ is a partition of this union.

12.6 Let b be any real number in the image of f, so there exists $a \in X$ such that $f(x) = b$. Let $V = f^{-1}(b)$ Then V is open in X: by Proposition 7.2 it is enough to show that for any $x \in V$ there exists an open set U_x of X with $x \in U_x \subseteq V$. This follows from the definition of locally constant, since $f(x) = b$ and there exists an open subset U_x of X containing x and such that $f|U_x$ takes the constant value $f(x) = b$ on U_x, so $U_x \subseteq V$.

Likewise for any point $y \in \mathbb{R}$ the set $f^{-1}(y)$ is open in X. But if $f^{-1}(y)$ is non-empty for any $y \neq b$ then we would get a partition $\{U, V\}$ of X, where

$$U = \bigcup_{y \in \mathbb{R} \setminus \{b\}} f^{-1}(y), \quad V = f^{-1}(b).$$

Hence $V = X$ and f is constant on X.

If $\mathbb{R}$ is replaced by any topological space in the above argument it is still valid.

12.7 This follows from the intermediate value theorem since we may show that if the polynomial function is written f, then $f(x)$ takes a different sign for x large and negative from its sign when x is large and positive, so its graph must cross the x-axis somewhere. Explicitly, we may as well assume that the polynomial is monic (has leading coefficient 1) say

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = x^n + g(x) \text{ say.}$$

Since $g(x)/x^n \to 0$ as $x \to \pm\infty$, there exists $\Delta \in \mathbb{R}$ such that $|g(x)/x^n| < 1$ for $|x| \geqslant \Delta$. Thus $f(x)/x^n = 1 + g(x)/x^n > 0$ for $|x| \geqslant \Delta$. This says that for x large enough in size (positive or negative) $f(x)$ has the same sign as x^n. But n is odd, so $f(x) < 0$ for x large and negative and $f(x) > 0$ for x large enough and positive.

12.8 Since $f([a, b]) \subseteq [a, b]$, we see that $f(a) \geqslant a$ and $f(b) \leqslant b$. Now define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - x$. then we have g is continuous since f is, and $g(a) \geqslant 0$, $g(b) \leqslant 0$. hence by the intermediate value theorem there is at least one point x in $[a, b]$ such that $g(x) = 0$, which says $f(x) = x$.

12.9 Define g as the hint suggests. Then g is continuous on $[0, 1]$ and

$$g(0)+g(1/n)+\ldots+g((n-1)/n) = f(0)-f(1/n)+f(1/n)-f(2/n)+\ldots f((n-1)/n)-f(1)$$
$$= f(0)-f(1) = 0.$$

Case 1 All the $g(i/n)$ are zero. Now if $g(i/n) = 0$ then $f(i/n) = f((i+1)/n)$ and the conclusion holds.

Case 2 For some $i = 0, 1, \ldots, (n-1)$ the values $g(i/n)$, $g((i+1)/n$ have opposite signs. But if $g(i/n)$, $g((i+1)/n)$ have opposite signs then by the intermediate value theorem there exists some $x \in [i/n, (i+1)/n]$ with $g(x) = 0$, so $f(x) = f(x+1/n)$.

12.10 Suppose for a contradiction that there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$ the set $f^{-1}(x)$ contains exactly two points.

The way to get a contradiction is reasonably clear geometrically, but the proof needs some organisation.

Choose some $x \in \mathbb{R}$ and let a, b be the two points in $f^{-1}(x)$. We may assume without loss of generality that $a < b$ (otherwise interchange their names). From standard properties of continuity of a real-valued function of a real variable (which will be generalised in Chapter 13), f is bounded on $[a, b]$ and attains its bounds there. Now the given condition means that f cannot be constant on the interval $[a, b]$, so either the maximum M of f on $[a, b]$ satisfies $M > f(a)$ or the minimum m satisfies $m < f(a)$.

Suppose that $M > f(a)$, and that f attains M at $c \in (a, b)$, i.e. $f(c) = M$. Choose some value d with $f(a) < d < M$. Since $f(b) = f(a)$, the intermediate value theorem tells us that there exist at least one value $x_1 \in (a, c)$ and at least one value $x_2 \in (c, b)$ such that $f(x_1) = d = f(x_2)$. Now choose some real number $\Delta > M$. We are given that $f(x) = \Delta$ for precisely two values of x. Since M is the maximum of f on $[a, b]$, neither of these values of x is in $[a, b]$. Suppose for example that $f(x) = \Delta$ for some $x < a$. Since d lies between $f(a)$ and Δ, the intermediate value theorem tells us that $f(x_3) = d$ for some $x_3 \in (x, a)$. This contradicts the assumption that $f^{-1}(d)$ contains exactly two points. A similar argument leads to the same contradiction if $f(x) = \Delta$ for some $x > b$.

If $m < f(a)$ an entirely similar pair of arguments again lead to a contradiction. Hence there is no such continuous function.

12.11 (a) This is false. For example let $X = Y = \mathbb{R}$ and let $A = B = \{0\}$. Then

$$X \setminus A = Y \setminus B = \mathbb{R} \setminus \{0\}$$

which is not connected, but $X \times Y \setminus (A \times B) = \mathbb{R}^2 \setminus \{(0, 0)\}$ which is path-connected and hence connected.

Note that common sense suggests (b) false, (c) true, since the conclusions of both are the same, but the hypotheses are stronger in (c). (This *proves* nothing, but it is suggestive.)

(b) This is false. For example let $X = \mathbb{R}$ and let $A = \{0, 1\}$, $B = (0, 1]$. Then both of $A \cap B = \{1\}$ and $A \cup B = [0, 1]$ are connected, but A is not connected.

(c) This is true. We shall prove it in the style of Definition 12.1. So let $f : A \to \{0, 1\}$ be continuous, where $\{0, 1\}$ has the discrete topology. Then $f|A \cap B$ is continuous, and since $A \cap B$ is connected, $f|A \cap B$ is constant, say with value c (where $c = 0$ or 1). Define $g : A \cup B \to \{0, 1\}$ by $g|A = f$, $g|B = c$. This is continuous by Exercise 10.7 (b), since each of A, B is closed in x and hence in $A \cup B$, and on the intersection $A \cap B$ the two definitions agree. But $A \cup B$ is connected, so g is constant. In particular this implies that $f : A \to \{0, 1\}$ is constant. So A is connected. Similarly B is connected.

12.12 Suppose that f, g are any two points in $\mathcal{C}[0, 1]$. We define a path $\lambda : [0, 1] \to \mathcal{C}[0, 1]$ by $\lambda(t) = tf + (1 - t)g$. Then for each $t \in [0, 1]$ the function $\lambda(t)$ is continuous hence in $\mathcal{C}[0, 1]$. Also, the function λ is continous since

$$d_\infty(\lambda(t), \lambda(s)) = \sup_{x \in [0, 1]} |(t - s)f(x) + (s - t)g(x)| \leqslant |t - s|(M_f + M_g),$$

where $|f(x)| \leqslant M_f$ and $|g(x)| \leqslant M_g$ for all $x \in [0, 1]$. Now given $\varepsilon > 0$, let $\delta = \varepsilon/(M_f + M_g)$, and we get $d_\infty(\lambda(t), \lambda(s)) < \varepsilon$ whenever $|t - s| < \delta$.

Finally, $\lambda(0) = g$ and $\lambda(1) = f$, so λ is a continuous path in $\mathcal{C}[0, 1]$ from g to f. Hence $\mathcal{C}[0, 1]$ is path-connected and so connected.

12.13 Let $y_1, y_2 \in Y$. Since f is onto, $f(x_1) = y_1$, $f(x_2) = y_2$ for some points $x_1, x_2 \in X$. Let $g : [0, 1] \to X$ be a continuous path in X from x_1 to x_2. Then $f \circ g : [0, 1] \to Y$ is a continuous path in Y from y_1 to y_2. So Y is path-connected.

12.14 There are various approaches. Here is one (possibly slightly unusual) which uses the previous exercise. Write Y for the annulus in the question, and let $X = [\sqrt{a}, \sqrt{b}] \times \mathbb{R} \subseteq \mathbb{R}^2$. Define $f : X \to Y$ by $f(x, y) = (c + x\cos y, d + x\sin y)$. First, $(c + x\cos y, d + x\sin y) \in Y$ since $(x\cos y)^2 + (x\sin y)^2 = x^2$, and we know $a \leqslant x^2 \leqslant b$ since $x \in [\sqrt{a}, \sqrt{b}]$. Also, f is continuous since the cosine and sine functions are continuous (and using Propositions 8.3 and 5.17). To see that f is onto, let $(x, y) \in Y$. Then $a \leqslant (x - c)^2 + (y - d)^2 \leqslant b$, say $(x - c)^2 + (y - d)^2 = r^2$ for some $r > 0$ such that $a \leqslant r^2 \leqslant b$. This gives $\left(\dfrac{x - c}{r}\right)^2 + \left(\dfrac{y - d}{r}\right)^2 = 1$, so there exists $\theta \in \mathbb{R}$ such that $\cos\theta = (x - c)/r$ and $\sin\theta = (y - d)/r$. Now we can see that $f(r, \theta) = (x, y)$. Since X is path-connected (any two points in X may be joined by a straight-line segment in X) it follows from Exercise 12.13 that Y is path-connected.

12.15 This follows from Exercise 9.14 (b), which says that a subset A is open and closed in X iff it has empty boundary. Since X is connected iff no proper non-empty subset is open and closed in X, it is connected iff every proper non-empty subset of X has non-empty boundary.

12.16 Suppose that for the given subsets, B has non-empty intersection with A and with $X \setminus A$. Suppose for a contradiction that $B \cap \partial A = \emptyset$. By Exercise 9.15 then $B \cap \mathring{A} \neq \emptyset$. Recalling that $\partial(X \setminus A) = \partial A$, the same argument shows that B has non-empty intersection with the interior of $X \setminus A$. But the interiors of A and $X \setminus A$ are open in X, so given that $B \cap \partial A = \emptyset$ by assumption, Exercise 9.16 shows that B is partitioned by its intersections with the interiors of A and $X \setminus A$. This contradicts connectedness of B, so $B \cap \partial A \neq \emptyset$.

12.17 Suppose that $f : A \cup B \to \{0,\, 1\}$ is continuous, where $\{0,\, 1\}$ has the discrete topology. Since A, B are connected, both $f|A$ and $f|B$ are constant, say with values $c_A,\, c_B \in \{0,\, 1\}$. Let $a \in A \cap \overline{B}$. Then $f^{-1}(c_A)$ is an open set containing a and hence some point $b \in B$. But then $f(b) = c_A$ and also $f(b) = c_B$ since $b \in B$. So $c_A = c_B$, and f is constant. Hence $A \cup B$ is connected.

12.18 As the hint suggests, let $t \notin X$ and consider the family of subsets $\mathcal{T}'$ of $Y = X \cup \{t\}$ consisting of $\emptyset$ together with $\{U \cup \{t\} : U \in \mathcal{T}\}$ where $\mathcal{T}$ is the topology on X. We prove first that $\mathcal{T}'$ is a topology for Y.

(T1) By definition $\emptyset \in \mathcal{T}'$, and $Y = X \cup \{t\} \in \mathcal{T}'$ since $X \in \mathcal{T}$.

(T2) Suppose that $V_1,\, V_2 \in \mathcal{T}'$. If either of $V_1,\, V_2$ is empty then $V_1 \cap V_2 = \emptyset$ and is in $\mathcal{T}'$ by definition. If $V_1 = U_1 \cup \{t\}$ and $V_2 = U_2 \cup \{t\}$ where $U_1,\, U_2 \in \mathcal{T}$ then $V_1 \cap V_2 = (U_1 \cap U_2) \cup \{t\}$ and $U_1 \cap U_2 \in \mathcal{T}$ so $V_1 \cap V_2$ is in $\mathcal{T}'$.

(T3) Suppose that $V_i \in \mathcal{T}'$ for all i in some indexing set I. If all the V_i are empty then so is their union, and is in $\mathcal{T}'$ by definition. Suppose that not all the V_i are empty. Then we may omit any empty sets since they do not contribute to the union - in other words we may assume that every V_i is non-empty. Hence for each $i \in I$ there is some $U_i \in \mathcal{T}$ such that $V_i = U_i \cup \{t\}$. Then

$$\bigcup_{i \in I} V_i = \left(\bigcup_{i \in I} U_i \right) \cup \{t\}.$$

Now $\mathcal{T}$ is a topology so $\bigcup_{i \in I} U_i \in \mathcal{T}$, and this shows that $\bigcup_{i \in I} V_i \in \mathcal{T}'$.

Next we show that $(X,\, \mathcal{T})$ is a subspace of $(Y,\, \mathcal{T}')$. We first show that the topology induced on X by $\mathcal{T}'$ is contained in $\mathcal{T}$. If $U = V \cap X$ for some $V \in \mathcal{T}'$ then either $V = \emptyset$ and $U = \emptyset \in \mathcal{T}$, or $V = U \cup \{t\}$ for some $U \in \mathcal{T}$, and then $V \cap X = U \in \mathcal{T}$. Secondly we show that $\mathcal{T}$ is contained in the topology on X induced by $\mathcal{T}'$. For if $U \in \mathcal{T}$ let $V = U \cup \{t\}$. Then $V \in \mathcal{T}'$, and $V \cap X = U$, so U is in the induced topology as required.

The last requirement, that $Y \setminus X$ is a single point, is clear from the construction.

12.19 The idea of this example is an infinite ladder where we kick away a rung at a time. Explicitly, let

$$V_n = ([0,\,\infty] \times \{0,\,1\}) \cup \bigcup_{i \in \mathbb{N},\, i \geqslant n} \{i\} \times [0,\,1].$$

Then it is clear that each V_n is path-connected hence connected, and that $V_n \supseteq V_{n+1}$ for each $n \in \mathbb{N}$. But

$$\bigcap_{n=1}^{\infty} V_n = [0,\,\infty] \times \{0,\,1\},$$

which is not connected.

1. Following the hint,let $x, y \in X$, and define $f(y) = d(x, y)$. Since X is connected, the same is true for $f[X]$, and therefore f takes every real value between $0 = f(x)$ and $f(y)$. In particular, for each a such that $0 \leq a \leq d(x, y)$, there is some point $z \in X$ such that $f(z) = d(x, z) = a$. By Rule 2 for cardinal numbers, we know that $|X|$ is greater than or equal to the cardinality of $f[X]$. Since the latter contains the open interval $(0, f(x))$ and the cardinality of the latter equals $|\mathbb{R}|$ by the proposition in `cardinals.pdf`, it follows that $|X| \geq |f[X]| \geq |\mathbb{R}|$.■

2. *(i)* The closure of $U = (0, 1)^2$ is $[0, 1]^2$, and therefore A satisfies $U \subset A \subset \overline{U}$. Therefore A is connected by Proposition 12.19 of Sutherland.■

(ii) If $S \neq T$ then

$$S \times \{1\} \quad = \quad A_S \ - \ U \ , \qquad T \times \{1\} \quad = \quad A_T \ - \ U$$

which implies that $A_S \neq A_T$. Therefore there is a 1–1 map from the set of subsets of $(0, 1)$ to $\mathcal{K}$ where $\mathcal{K}$ is the set of connected subsets of $\mathbb{R}^2$. Since $\mathbb{R}$ and $(0, 1)$ have the same cardinality, it follows that $|\mathcal{P}(\mathbb{R})| \leq |\mathcal{K}|$. On the other hand, by construction we have $\mathcal{K} \subset \mathcal{P}(\mathbb{R}^2)$, which yields $|\mathcal{K}| \leq |\mathcal{P}(\mathbb{R}^2)|$. Since $|\mathbb{R}^2| = |\mathbb{R}|$ by Rule 6, we can use Rule 7 to conclude that $|\mathcal{K}| \leq |\mathcal{P}(\mathbb{R}^2)| = |\mathcal{P}(\mathbb{R})$, and the desired identity $|\mathcal{K}| = |\mathcal{P}(\mathbb{R})$ now follows from the Schröder-Bernstein Theorem.■

3. *Notational convention for this exercise.* The standard notation for point set theory contains a built-in conflict which is usually not a problem, but for this exercise it can lead to confusion; namely, if a and b are real numbers then (a, b) can mean either an open interval or a point in $\mathbb{R}^2$. In this exercise, we shall eliminate this ambiguity by using $(a; b)$, $[a; b]$, *etc.* to denote intervals.

Let C be a nonempty connected subset of $\mathbb{R}$, so that $u, v \in C$ and $u < v$ imply $[u, v] \subset C$. Let $a(C)$ be the greatest lower bound of C if C has a lower bound, and let $a(C) = -\infty$ otherwise. Similarly, let $b(C)$ be the least upper bound of C if C has an upper bound, and let $b(C) = +\infty$ otherwise. Let p be some point of C.

CLAIM: The set C is an interval whose lower endpoint is $a(C)$ and whose upper endpoint is $b(C)$. — Since $a(C)$ and $b(C)$ are lower and upper bounds for C, it follows that C is contained in the set of all points x satisfying $a(C) \leq x \leq b(C)$, with the convention that there is strict inequality if $a(C) = -\infty$ or $b(C) = +\infty$. If $a(C) < b(C)$ and $p \in C$ lies between these values, take sequences $\{u_n\}$ and $\{v_n\}$ in C such that $u_n \to a(C)$, $v_n \to b(C)$, and $u_n < p < v_n$ for all n. By the assertion in the first sentence of the first paragraph, then $[u_n; v_n] \subset C$ for all n, and therefore the union of these subsets, which is the set of all x such that $a(C) \leq x \leq b(C)$, is contained in C. Combining this with the previous paragraph, we have

$$\{x \in \mathbb{R} \mid a(C) \ < \ x \ < \ b(C)\} \quad \subset \quad C \quad \subset \quad \{x \in \mathbb{R} \mid a(C) \leq x \leq b(C)\}$$

with the previous conventions if $a(C) = -\infty$ or $b(C) = +\infty$. For each pair of values $a(C)$ and $b(C)$ there are up to four choices for C, depending upon whether or not $a(C)$ or $b(C)$ belong to C.

The preceding discussion reduces the solution of the exercise to finding the cardinality of the set of intervals in $\mathbb{R}$. We shall do this by splitting the family of intervals into nine pairwise disjoint subfamilies:

(1) $\mathcal{F}_1$ consists only of the real line $\mathbb{R} = (-\infty; +\infty)$.

(2) $\mathcal{F}_2$ consists of the open half-lines $(-\infty; b)$ where $b \in \mathbb{R}$.

(3) $\mathcal{F}_3$ consists of closed half-lines $(-\infty; b]$ where $b \in \mathbb{R}$.

(4) $\mathcal{F}_4$ consists of the open half-lines $(a; +\infty)$ where $a \in \mathbb{R}$.

(5) $\mathcal{F}_5$ consists of the closed half-lines $[a; +\infty)$ where $a \in \mathbb{R}$.

(6) $\mathcal{F}_6$ consists of the open intervals $(a; b)$ where $a; b \in \mathbb{R}$.

(7) $\mathcal{F}_7$ consists of the half-open intervals $(a; b]$ where $a, b \in \mathbb{R}$.

(8) $\mathcal{F}_8$ consists of the half-open intervals $[a; b)$ where $a, b \in \mathbb{R}$.

(9) $\mathcal{F}_9$ consists of the closed intervals $[a; b]$ where $a, b \in \mathbb{R}$.

By Rule 6 and Proposition 1 in `cardinals.pdf` it will suffice to construct 1–1 mappings φ_j from $\mathcal{F}_j$ to some $\mathbb{R}^k$ for each j, and we shall do so case by case. Take φ_1 to be the map sending the unique element of $\mathcal{F}_1$ to $0 \in \mathbb{R}$, take φ_2 and φ_3 to be the maps into $\mathbb{R}$ sending a half-line with a finite right hand endpoint to its right hand endpoint b, and take φ_4 and φ_5 to be the maps into $\mathbb{R}$ sending a half-line with a finite left hand endpoint to its left hand endpoint a. If $6 \le j \le 9$, take φ_j to be the map into $\mathbb{R}^2$ sending an interval with finite left hand endpoint a and finite right hand endpoint b to $(a, b) \in \mathbb{R}^2$ (recall the conventions regarding (a, b) at the beginning of the exercise).

By construction each of these mappings into some $\mathbb{R}^k$ is 1–1, and therefore Proposition 1 and Rule 6 in `cardinals.pdf` implies that the cardinality of the set of connected subsets of $\mathbb{R}$ is less than or equal to $|\mathbb{R}|$. Since the cardinality of the set of open half-lines $(a, +\infty)$ is clearly equal to $|\mathbb{R}|$, it follows that the cardinality of the set of connected subsets of $\mathbb{R}$ is also greater than or equal to $|\mathbb{R}|$, so by the Schröder-Bernstein Theorem the cardinality of the set of connected subsets of $\mathbb{R}$ is in fact equal to $|\mathbb{R}|$.

We can use the results on cardinalities of connected subsets to give another proof that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic provided we can show the following:

CLAIM: *If $f : X \to Y$ is a homeomorphism of topological spaces, then it induces a 1–1 correspondence from the family of connected subsets in X to the corresponding family in Y. In fact, $A \subset X$ is connected if and only if $f[A]$ is connected in Y.*

Proof of Claim. We know that if A is connected then so is $f[A]$. Conversely, if $f[A]$ is connected and h is inverse to f, then $A = h\,[\,f[A]\,]$ is also connected.∎

Since the cardinalities for the connected subsets of $\mathbb{R}$ and $\mathbb{R}^2$ are different, this result shows that the two spaces cannot be homeomorphic.∎

4. (i) If X is a discrete space, then a base for the topology is given by the one point subsets $\{x\}$ where x runs through all the points of X; since these one point subsets are both open and closed, the topology consists entirely of open and closed subsets and hence X is totally disconnected.

As indicated in the hint, a subset A of a discrete space has no limit points, for if $x \in X$ and $U = \{x\}$ then $U - \{x\} = \emptyset$, so that $A \cap (U - \{x\}) = \emptyset$. Now if S is the set of all points x in $\mathbb{R}$ such that $x = 0$ or $x = 1/n$ for some positive integer n, then S is not discrete because 0 is a limit point of S, but S is totally disconnected; specifically, a base of subsets which are both open and closed in S is given by the one point sets $\{1/n\}$ and the neighborhoods $N_\varepsilon(0) \cap S$ for $\varepsilon = 2/(n+1)$ because $N_\varepsilon(0) \cap S$ is also the intersection of S with both the open interval $(-1, 1/n)$ and the closed interval $[-1, 1/(n+1)]$.∎

(ii) As noted in the hint, for each positive integer n the sets

$$\left(q - \tfrac{1}{n}\sqrt{2},\ \ q + \tfrac{1}{n}\sqrt{2}\right) \cap \mathbb{Q} \qquad \text{and} \qquad \left[q - \tfrac{1}{n}\sqrt{2},\ \ q + \tfrac{1}{n}\sqrt{2}\right] \cap \mathbb{Q}$$

are equal; this is true because $\left[q - \frac{1}{n}\sqrt{2},\ \ q + \frac{1}{n}\sqrt{2}\right]$ is obtained from $\left(q - \frac{1}{n}\sqrt{2},\ \ q + \frac{1}{n}\sqrt{2}\right)$ by adjoining two endpoints which are irrational numbers, and it follows that the intersection of these subsets with $\mathbb{Q}$ is both open and closed in $\mathbb{Q}$. If we denote this subset of $\mathbb{Q}$ by $M_n(q)$ then for each $\varepsilon > 0$ there is some n such that $M_n(q) \subset (q - \varepsilon, q + \varepsilon)$, and therefore the subsets $M_n(q)$ — where n runs through all positive integers and q runs through all rationals — form a base for the topology of $\mathbb{Q}$ whose subsets are both open and closed, and hence $\mathbb{Q}$ is totally disconnected. To see that every point of $\mathbb{Q}$ is a limit point of $\mathbb{Q}$, it suffices to consider the sequences $\{q + \frac{1}{n}\}$ for each rational number q.■

(*iii*) Since X and Y are totally disconnected, there are bases $\mathcal{B}_X$ and $\mathcal{B}_Y$ for the topologies on X and Y which consist of subsets that are both open and closed. By definition and a previous exercise, we know that if $U \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$ then $U \times V$ is both open and closed in $X \times Y$. Since product sets of this type form a basis for the product topology on $X \times Y$, we have shown that the product topology has a base of subsets which are both open and closed, and therefore $X \times Y$ is totally disconnected.■

5. (*i*) Following the hint, define a binary relation on U such that $y \sim z$ if and only if either $y = z$ or there is a regular piecewise smooth curve $\gamma : [a, b] \to U$, where $[a, b]$ is some closed interval, such that $\gamma(a) = y$ and $\gamma(b) = z$. We need to show this is an equivalence relation.

By definition, the relation is reflexive, and it is symmetric because the reverse curve $\beta : [-b, -a] \to U$ with $\beta(t) = \gamma(-t)$ is a regular piecewise smooth curve with $\beta(-b) = z$ and $\beta(-a) = y$. Transitivity follows from the concatentation construction used in the proof (in the text and lecture notes) that U is arcwise connected.

If $z \in U$ and $N_r(z; \mathbb{R}^n) \subset U$, then $z \sim w$ for every point in $N_r(z; \mathbb{R}^n)$ because the latter is convex and hence contains the straight line (regular smooth) curve $\alpha(s) = z + s(w - z)$. This means that the equivalence class of a point is open in U. Therefore the equivalence classes of the binary relation partition U into pairwise disjoint open subsets, one for each equivalence class. If $V \subset U$ is the equivalence class of V, this means not only that V is open, but also that its complement — which is the union of all other equivalence classes (if there are any others) — is also open. Therefore V is both open and closed in U, and by connectedness we must have $V = U$. But this means that if p and q are points of the open set U satisfying the conditions at the beginning of this exercise, then there is a regular piecewise smooth curve $\gamma : [a, b] \to U$ such that $\gamma(a) = p$ and $\gamma(b) = q$.■

Note. The same general considerations imply that two points in an open connected subset of $\mathbb{R}^n$ can joined by regular piecewise smooth curve which have additional properties. For example, if we define a *broken line curve* such that on each of the subintervals $[t_{k-1}, t_k]$ it is a closed segment with a linear parametrization

$$\gamma(t) \;=\; \gamma(t_{k-1}) \;+\; \frac{t - t_{k-1}}{t_k - t_{k-1}} \cdot \big(\gamma(t_k) \;-\; \gamma(t_{k-1})\big) \qquad \text{for} \quad t \;\in\; [t_{k-1}, t_k]$$

then it is a straightforward exercise to modify the preceding argument to prove that each pair of points in U can be joined by a broken line curve in U. In a different direction, one can also prove that each pair of points in U can be joined by a curve that is everywhere regular and has infinitely differentiable coordinate functions, but this requires some additional input. Details for this result (and the previous one on broken lines) appear in the course directory file `nicecurves.pdf`.■

(*ii*) Once again follow the hint. The binary relation is reflexive since $f(y) = f(y)$, reflexive since $f(y) = f(z)$ implies $f(z) = f(y)$, and transitive since $f(y) = f(z)$ and $f(z) = f(w)$ implies $f(y) = f(w)$. If $z \in U$ and $N_r(z; \mathbb{R}^n) \subset U$, then a standard result on partial differentiation implies that a function which satisfies the vanishing condition on partial derivatives will be constant on $N_r(z;\mathbb{R}^n)$. This means that the equivalence class of a point is open in U. Therefore the equivalence classes of the binary relation partition U into pairwise disjoint open subsets, one for each equivalence class. If $V \subset U$ is the equivalence class of V, this means not only that V is open, but also that its complement — which is the union of all other equivalence classes (if there are any others) — is also open. Therefore V is both open and closed in U, and by connectedness we must have $V = U$. But this means that f is constant on U.■

Note. The preceding argument can be put into the following abstract form: *Suppose that V is a connected open subset of $\mathbb{R}^n$ and $f : V \to Y$ is a function which is locally constant; in other words, each point of V has an open neighborhood on which f is constant. The f is a constant function.*■