SOLUTION TO SUTHERLAND, 12.12(c)

Proof that A, B closed in $X = A \cup B$ with X and $A \cap B$ connected $\Rightarrow$ A, B connected.

It suffices to show A is connected (switch A & B to get the other conclusion).

Suppose $A = C_1 \cup C_2$, C_i nonempty closed & disjoint. Then $A \cap B$ connected $\Rightarrow$ either $A \cap B \subseteq C_1$ or $A \cap B \subseteq C_2$. Without loss of generality we may assume $A \cap B \subseteq C_1$.

Let $B_1 = B \cup C_1$, a closed subset of X. We claim $B_1 \cap C_2 = \phi$, so that $X = A \cup B = B_1 \cup C_2$ is a disjoint union of these closed sets. If A is not connected, then $C_2 \neq \phi$, $C_1 \neq \phi$ imply X is not connected — contradiction. Therefore A must be connected.