SOLUTIONS TO EXERCISES FOR

MATHEMATICS 145A — Part 6

Winter 2014

13. The Compact spaces

Exercises from Sutherland

`See the next six pages.`

Solutions to Chapter 13 exercises

13.1 Suppose that the space X has the indiscrete topology. Then the only open sets in X are $\emptyset$, X. So any open cover of X must contain the set X, and $\{X\}$ is a finite subcover.

13.2 Let the space X have the discrete topology. First suppose that X is finite. Then there are only a finite number of distinct subsets of X, so any open cover of X already is a finite subcover of itself, provided the sets in the cover are distinct. Hence X is compact.

But if X is infinite then it has the open cover $\{\{x\} : x \in X\}$ which clearly has no finite subcover. So if X is compact it must be finite.

13.3 Suppose that $\mathcal{U}$ is an open cover of $A \cup B$. Then in particular $\mathcal{U}$ is an open cover of A, so there is a finite subfamily $\mathcal{U}_A$ of $\mathcal{U}$ which covers A. Similarly there is a finite subfamily $\mathcal{U}_B$ of $\mathcal{U}$ which covers B. Then $\mathcal{U}_A \cup \mathcal{U}_B$ is a finite subcover of $A \cup B$, and this proves that $A \cup B$ is compact.

13.4 Throughout this question we use the characterization of compact subsets in Euclidean spaces: $C \subseteq \mathbb{R}^n$ is compact iff it is bounded and closed in $\mathbb{R}^n$.

(i) This set is not closed in $\mathbb{R}$, hence not compact.

(ii) This set is not bounded so not compact.

(iii) This set is not closed in $\mathbb{R}$ hence not compact (for example $1/\sqrt{2}$ is in its closure in $\mathbb{R}$ but not in the set).

(iv) This set is bounded and closed in $\mathbb{R}^2$ and hence compact. (It is clearly bounded. To see that it is closed in $\mathbb{R}^2$ we note that it is $f^{-1}(1)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is the continuous function defined by $f(x, y) = x^2 + y^2$.)

(v) This set is bounded and closed in $\mathbb{R}^2$ and hence compact. (It is clearly bounded. To see that it is closed in $\mathbb{R}^2$ we note that it is $f^{-1}([0, 1])$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is the continuous function defined by $f(x, y) = |x| + |y|$.)

(vi) This set in not closed in $\mathbb{R}^2$ hence not compact. (For example the point $(1, 0)$ is in its closure but not in the set.)

(vii) This set is not bounded hence not compact. For given any $n \in \mathbb{N}$ the point $(1/n, n)$ is in the set.

13.5 Suppose that $\mathcal{U}$ is any open cover of $(X, \mathcal{T})$. Since $\mathcal{T} \subseteq \mathcal{T}'$, each set in $\mathcal{U}$ is in $\mathcal{T}'$, hence $\mathcal{U}$ is an open cover of $(X, \mathcal{T}')$ as well. But $(X, \mathcal{T}')$ is compact, so there is a finite subcover. This proves that $(X, \mathcal{T})$ is compact.

13.6 First suppose that X is compact. Let $\{V_i : i \in I\}$ be an indexed family with the property that $\bigcap_{j\in J} V_j$ is non-empty for every finite subset $J \subseteq I$ (this is called the *finite intersection property*). Suppose for a contradiction that $\bigcap_{i\in I} V_i = \emptyset$. Then the family $\{X \setminus V_i : i \in I\}$ is an open cover for X, for each $X \setminus V_i$ is open since V_i is closed, and

$$\bigcup_{i\in I}(X \setminus V_i) = X \setminus \bigcap_{i\in I} V_i = X \setminus \emptyset = X.$$

Since X is compact, this open cover has a finite subcover, so there is some finite subset $J \subset I$ such that $X \setminus \bigcap_{j\in J} V_j = \bigcup_{j\in J} X \setminus V_j = X$, so $\bigcap_{j\in J} V_j = \emptyset$, which contradicts the finite intersection property. Hence $\bigcap_{i\in I} V_i \neq \emptyset$.

Conversely suppose that X has the property described in this exercise, and let $\{U_i : i \in I\}$ be an open cover of X. Write $V_i = X \setminus U_i$. Then each V_i is closed in X, and

$$\bigcap_{i\in I} V_i = \bigcap_{i\in I}(X \setminus U_i) = X \setminus \bigcup_{i\in I} U_i = X \setminus X = \emptyset,$$

so there must be some finite subset $J \subseteq I$ such that $\bigcap_{j\in J} V_j = \emptyset$. Then

$$\bigcup_{j\in J} U_j = \bigcup_{j\in J}(X \setminus V_j) = X \setminus \bigcap_{j\in J} V_j = X \setminus \emptyset = X.$$

This says that $\{U_j : j \in J\}$ is a finite subcover of $\{U_i : i \in I\}$. Hence X is compact.

13.7 This is immediate since any finite subset of a space is compact.

13.8 First suppose that $X \subset \mathbb{R}$ is unbounded. Let $f : X \to \mathbb{R}$ be the inclusion function. Then f is continuous and unbounded.

Secondly suppose that $X \subseteq \mathbb{R}$ is not closed in $\mathbb{R}$. Let $c \in \overline{X} \setminus X$, and define $f : X \to \mathbb{R}$ by $f(x) = 1/(|x-c|)$. Then f is continuous, since for each $x \in X$ we have $|x - c| \neq 0$. But f is not bounded, since $|x-c|$ gets arbitrarily small so $1/(|x-c|)$ gets arbitrarily large: explicitly, let $\Delta \in \mathbb{R}$ with $\Delta > 0$. Then $1/\Delta > 0$, so since $c \in \overline{X}$ there exists $x \in X$ with $|x - c| < 1/\Delta$ and hence $f(x) > \Delta$.

13.9 We proceed as in the previous exercise. Suppose first that $X \subseteq \mathbb{R}$ is unbounded. Define $f : X \to \mathbb{R}$ by $f(x) = 1/(1+|x|)$. Then the lower bound of f is 0. For $f(x) > 0$ for all $x \in X$, but for any $\delta > 0$, there exists $x \in X$ such that $1 + |x| > 1/\delta$, and then $f(x) < \delta$. But f does not attain its lower bound 0 since $f(x) > 0$ for all $x \in X$.

Secondly suppose that $X \subseteq \mathbb{R}$ is not closed. Let $c \in \overline{X} \setminus X$. Define $f : X \to \mathbb{R}$ by $f(x) = |x - c|$. Then the lower bound of f is 0 since for any $\delta > 0$ there exists $x \in X$ with $|x - c| < \delta$. But this lower bound is not attained, since $f(x) > 0$ for all $x \in X$.

13.10 Since C and C' are compact subspaces of a Hausdorff space X they are closed in X (by Proposition 13.12). Hence $C \cap C'$ is closed in X. But then $C \cap C'$ is also closed in C (by Exercise 10.5), and since C is compact so is $C \cap C'$ (by Proposition 13.20).

13.11 This follows immediately from Exercise 13.6, since the family $\{V_n : n \in \mathbb{N}\}$ has the finite intersection property - the intersection of any finite subfamily $\{V_{n_1}, V_{n_2}, \dots, V_{n_r}\}$ is V_N where $N = \max\{n_1, n_2, \dots, n_r\}$, and V_N is non-empty. Hence since X is compact, Exercise 13.6 tells us that $\bigcap_{n=1}^{\infty} V_n$ is non-empty.

13.12 As the hint suggests, we consider the sequence of sets $W_n = V_n \cap (X \setminus U)$. Since the V_n are nested, so are the W_n. Also, each W_n is closed in X since V_n is closed in X and U is open in X so $X \setminus U$ is closed in X. Suppose for a contradiction that there is no integer n such that $V_n \subseteq U$. Then each set W_n is non-empty, so by the previous question $\bigcap_{n=1}^{\infty} W_n$ is non-empty. But

$$\bigcap_{n=1}^{\infty} W_n = \left(\bigcap_{n=1}^{\infty} V_n \right) \cap (X \setminus U) = V_\infty \cap (X \setminus U)$$

and this is empty since $V_\infty \subseteq U$. This contradiction shows that $V_n \subseteq U$ for some integer n.

13.13 (a) We show that the X_n form a nested sequence of closed subsets of X. Note that $X_1 = f(X_0) = f(X) \subseteq X_0$. Suppose inductively that $X_n \subseteq X_{n-1}$ for some integer $n \geqslant 1$. Then $f(X_n) \subseteq f(X_{n-1})$, which says $X_{n+1} \subseteq X_n$. So by induction $X_n \subseteq X_{n-1}$ for all $n \in \mathbb{N}$. Also, since X is compact and f is continuous, $X_1 = f(X)$ is compact (by Proposition 13.15). Suppose that X_n is compact for some integer $n \geqslant 0$. Then since f is continuous, $X_{n+1} = f(X_n)$ is also compact (by Proposition 13.15). By induction, X_n is compact for all integers $n \geqslant 0$. But X is Hausdorff, so each X_n is closed in X. Also, each X_n is non-empty by inductive construction. Now by Exercise 13.11, $A = \bigcap_{n=0}^{\infty} X_n$ is nonempty.

(b) The inclusion $f(A) \subseteq A$ is straightforward: if $a \in A$ then $a \in X_n$ for any integer $n \geqslant 0$, so $f(a) \in f(X_n) = X_{n+1}$ for every integer $n \geqslant 0$. But since $X_0 \supseteq X_1 \supseteq \dots \supseteq X_n \supseteq \dots$, we have $\bigcap_{n=1}^{\infty} X_n = \bigcap_{n=0}^{\infty} X_n$, and we see that $f(a) \in A$.

To prove the opposite inclusion we follow the hint, and for any $a \in A$ we let $V_n = f^{-1}(a) \cap X_n$. Since the X_n are nested. so are the V_n. Since X is Hausdorff, $\{a\}$ is closed in A, hence since f is continuous, $f^{-1}(a)$ is a closed subset of X by Proposition 9.5. Also, each X_n is closed in X as above. So each V_n is closed in X. Moreover, for any integer $n \geqslant 0$,we know $a \in X_{n+1} = f(X_n)$ so there exists $x \in X_n$ such that $f(x) = a$. This says that $V_n = f^{-1}(a) \cap X_n$ is non-empty. Now by Exercise 13.11, $\bigcap_{n=0}^{\infty} V_n$ is non-empty. Let b be a point in this set. Then $b \in V_n = f^{-1}(a) \cap X_n$ for all integers $n \geqslant 0$. Now $b \in f^{-1}(a)$ says that $f(b) = a$, and $b \in X_n$ for all integers $n \geqslant 0$ says that $b \in \bigcap_{n=0}^{\infty} X_n = A$. So $A \subseteq f(A)$. We have now proved that $f(A) = A$.

13.14 Following the hint, define $g : X \to \mathbb{R}$ by $g(x) = d(f(x), x)$. So g is the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times 1} X \times X \xrightarrow{d} \mathbb{R},$$

where Δ is the diagonal map. Now Δ, $f \times 1$ and d are continuous (by Propositions 10.13 and 10.12, and Exercise 5.17) so their composition g is continuous. Since X is compact, g attains its lower bound, say c, on X, (by Corollary 13.18) and we must have $c > 0$ since for every $x \in X$ we are given $f(x) \neq x$ so $d(f(x), x) > 0$. Hence $d(f(x), x) \geqslant c > 0$ for all $x \in X$.

13.15 The exercise gives the procedure for constructing the sequences (a_n), (b_n). The sequence (a_n) converges to some real number c since it is monotonic increasing and bounded above (for example by b_1). Likewise, since (b_n) is monotonic decreasing and bounded below (for example by a_1) it too converges to some real number d. Note that for any $m \geqslant n$ we have $a_n \leqslant a_m \leqslant b_m$, so in the limit as $m \to \infty$, $a_n \leqslant d$. This is true for all $n \in \mathbb{N}$, so in the limit as $n \to \infty$, $c \leqslant d$. Now for any $n \in \mathbb{N}$ we have $a_n \leqslant c \leqslant d \leqslant b_n$. Since $b_n - a_n = (b-a)/2^n$, also $d - c \leqslant (b-a)/2^n$. This gives $c = d$. But $c \in U$ for some $U \in \mathcal{U}$, and U is open in $\mathbb{R}$, so there exists $\varepsilon > 0$ such that $(c - \varepsilon, c + \varepsilon) \subseteq U$. Hence for large enough n, the interval $[a_n, b_n] \subseteq U$ since $c \in [a_n, b_n]$ and $b_n - a_n = (b-a)/2^n$. This contradicts the construction. So $\mathcal{U}$ must have a finite subcover after all.

13.16 Lipschitz equivalent metrics d_1, d_2 on a set X satisfy $hd_2(x, y) \leqslant d_1(x, y) \leqslant kd_2(x, y)$ for some positive constants h, k and all $x, y \in X$. Given $\varepsilon > 0$, let $\delta = h\varepsilon$. Then whenever $d_1(x, y) < \delta$ we have $d_2(x, y) \leqslant d_1(x, y)/h < \varepsilon$, for all $x, y \in X$. So the identity map of X is uniformly (d_1, d_2)-continuous. The proof that it is also uniformly (d_2, d_1)-continuous is entirely similar.

13.17 Suppose that $\mathcal{T}_1$, $\mathcal{T}_2$ are topologies on a set X, such that $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and that $(X, \mathcal{T}_1)$ is Hausdorff and $(X, \mathcal{T}_2)$ is compact. Consider the identity function from $(X, \mathcal{T}_2)$ to $(X, \mathcal{T}_1)$. This is continuous since $\mathcal{T}_1 \subseteq \mathcal{T}_2$, and it is certainly one-one onto. Since $(X, \mathcal{T}_1)$ is Hausdorff and $(X, \mathcal{T}_2)$ is compact, it follows from the inverse function theorem Proposition 13.26 that the identity function is a homeomorphism. So the identity function from $(X, \mathcal{T}_1)$ to $(X, \mathcal{T}_2)$ is also continuous. This says that $\mathcal{T}_2 \subseteq \mathcal{T}_1$. So $\mathcal{T}_1 = \mathcal{T}_2$ as required.

In particular if $X = [0, 1]$ and $\mathcal{T}_1$ is a Hausdorff topology on $[0, 1]$ which is contained in the Euclidean topology $\mathcal{T}_2$, then since we know that $([0, 1], \mathcal{T}_2)$ is compact, the first part of the exercise shows that $\mathcal{T}_1 = \mathcal{T}_2$. So $\mathcal{T}_1$ is not strictly coarser than the Euclidean topology $\mathcal{T}_2$.

13.18 Suppose for a contradiction that there is no point $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{F}$. Then for any $x \in X$ there is a function $f_x \in \mathcal{F}$ such that $f_x(x) \neq 0$, so $f_x(x) > 0$ by (i). By continuity of f_x, there is an open set $U(x) \ni x$ such that $f_x(y) > 0$ for all $y \in U(x)$. Now $\{U(x) : x \in X\}$ is an open cover of X, and X is compact, hence there is a finite subcover, say $\{U(x_1), U(x_2), \ldots, U(x_r)\}$. Let $f = f_{x_1} + f_{x_2} + \ldots + f_{x_r}$. Then by iterating (ii) we see that $f \in \mathcal{F}$. But any $x \in X$ is in $U(x_i)$ for some $i \in \{1, 2, \ldots, r\}$ so $f_{x_i}(x) > 0$ while $f_{x_j}(x) \geqslant 0$ for all $j \in \{1, 2, \ldots, r\}$. This gives $f(x) > 0$, and this is true for all $x \in X$, contradicting (iii). The result follows.

13.19 Let X be a compact Hausdorff space, let $W \subseteq X$ be closed in X and let $y \in X \setminus W$. Since W is a closed subset of a compact space, W is compact (by Proposition 13.20). For any $w \in W$, by the Hausdorff condition there exist disjoint open subsets U_w, V_w of X such that $y \in U_w$, $w \in V_w$. Now $\{V_w : w \in W\}$ is an open cover of the compact subset W, so there is a finite subcover $\{V_{w_1}, V_{w_2}, \ldots, V_{w_r}\}$. Put

$$U = \bigcap_{i=1}^{r} U_{w_i}, \quad V = \bigcup_{i=1}^{r} V_{w_i}.$$

Then U, V are open in X. Also, $W \subseteq V$ since $\{V_{w_1}, V_{w_2}, \ldots, V_{w_r}\}$ is a cover of W. Next, $y \in U$ since $y \in U_{w_i}$ for all $i \in \{1, 2, \ldots, r\}$. Finally, U, V are disjoint, since for any point $v \in V$ we have $v \in V_{w_i}$ for some $i \in \{1, 2, \ldots, r\}$, so since U_{w_i} and V_{w_i} are disjoint, $v \notin U_{w_i}$ hence $v \notin U$. This proves that X is regular.

The proof that X is normal is very similar. Suppose now that W, Y are disjoint closed subsets of X. From the first part, for each $y \in Y$ there exist disjoint open subsets U_y, V_y of X such that $y \in U_y$, $W \subseteq V_y$. Now $\{U_y : y \in Y\}$ is an open cover of Y, and Y is compact (by Proposition 13.20) so there is a finite subcover $\{U_{y_1}, U_{y_2}, \ldots, U_{y_s}\}$. Put

$$U = \bigcup_{j=1}^{s} U_{y_j}, \quad V = \bigcap_{j=1}^{s} V_{y_j}.$$

Then U, V are open in X. Also, $Y \subseteq U$ since $\{U_{y_1}, U_{y_2}, \ldots, U_{y_s}\}$ is a cover for Y. Next, $W \subseteq V$ since $W \subseteq V_{y_j}$ for all $j \in \{1, 2, \ldots, s\}$. Finally, U and V are disjoint, since if $u \in U$ then $u \in U_{y_j}$ for some $j \in \{1, 2, \ldots, s\}$, so $u \notin V_{y_j}$ since U_{y_j} and V_{y_j} are disjoint, so $u \notin V$. Hence X is normal.

13.20 (a) We show that $X \setminus p_X(W)$ is open in X using Proposition 7.2, that is by proving that if $x \in X \setminus p_X(W)$ then there is some open subset U of X such that $x \in U \subseteq X \setminus p_X(W)$.

So let $x \in X \setminus p_X(W)$. This means there is no $y \in Y$ such that $(x, y) \in W$. So $(x, y) \notin W$ for any $y \in Y$. Now W is closed in $X \times Y$, so (x, y) is in the set $(X \times Y) \setminus W$ which is open in $X \times Y$. Hence by definition of the product topology, there exist open subsets U_y, V_y of X, Y respectively such that $(x, y) \in U_y \times V_y \subseteq (X \times Y) \setminus W$. Now $\{V_y : y \in Y\}$ is an open cover of Y, and Y is compact, so there exists a finite subcover $\{V_{y_1}, V_{y_2}, \ldots, V_{y_r}\}$. Put

$$U = \bigcap_{i=1}^{r} U_{y_i}.$$

Then U is open in X, and $x \in U$ since $x \in U_{y_i}$ for each $i \in \{1, 2, \ldots, r\}$. Also, $U \subseteq X \setminus p_X(W)$, since if $x' \in U$ then given any $y \in Y$ we know that $y \in V_{y_i}$ for some $i \in \{1, 2, \ldots, r\}$, so from $x' \in U_{y_i}$ and $(U_{y_i} \times V_{y_i}) \cap W = \emptyset$ we get $(x', y) \notin W$. Since this is true for all $y \in Y$ it follows that $x' \notin p_X(W)$. This proves that $X \setminus p_X(W)$ is open in X hence $p_X(W)$ is closed in X.

(b) A suitable example is given by Exercise 10.15(b).

13.21 First, following the hint we prove that $f^{-1}(V) = p_X(G_f \cap p_Y^{-1}(V))$ for any subset $V \subseteq Y$. For if $x \in f^{-1}(V)$ then $f(x) \in V$, and $x = p_X(x, f(x))$, where $(x, f(x)) \in G_f$ and also $(x, f(x)) \in p_Y^{-1}(V)$ since $p_Y(x, f(x)) = f(x) \in V$.

Now suppose that $x \in p_X(G_f \cap p_Y^{-1}(V))$, so there exists $y \in Y$ such that $(x, y) \in G_f \cap p_Y^{-1}(V)$. Now $(x, y) \in G_f$ says that $y = f(x)$, and $(x, y) \in p_Y^{-1}(V)$ says that $y \in V$, so $f(x) \in V$ and $x \in f^{-1}(V)$ as required.

Still following the hint we apply the above to a closed subset $V \subseteq Y$. The graph G_f is given to be closed in $X \times Y$ and $p_Y^{-1}(V)$ is closed in $X \times Y$ (by Proposition 9.5) since V is closed in Y. So $G_f \cap p_Y^{-1}(V)$ is closed in $X \times Y$, and since also Y is compact, by Exercise 13.20 (a), $p_X(G_f \cap p_Y^{-1}(V))$ is closed in X. This tells us that $f^{-1}(V)$, which equals $p_X(G_f \cap p_Y^{-1}(V))$, is closed in X whenever V is closed in Y, so f is continuous by Proposition 9.5 .

13.22 We note that when $U' \in \mathcal{T}'$ with $U' = V \cup \{\infty\}$ for some $V \subseteq X$ such that $X \setminus V$ is closed in X (and compact) then $V \in \mathcal{T}$.

We check first that $\mathcal{T}'$ is a topology on X'.

(T1) $\emptyset \in \mathcal{T} \subseteq \mathcal{T}'$. Also, $X' = X \cup \{\infty\}$ and $X \setminus X = \emptyset$ is closed in X and compact, so $X' \in \mathcal{T}'$.

(T2) Suppose $U', V' \in \mathcal{T}'$. If $U', V' \in \mathcal{T}$ then $U' \cap V' \in \mathcal{T} \subseteq \mathcal{T}'$. If $U' \in \mathcal{T}$ and $V' = V \cup \{\infty\}$ for some $V \in \mathcal{T}$ then $U' \cap V' = U' \cap V \in \mathcal{T} \subseteq \mathcal{T}'$. A similar argument works when $U' = U \cup \{\infty\}$ for some $U \in \mathcal{T}$ and $V' \in \mathcal{T}$. Suppose finally that $U' = U \cup \{\infty\}$, $V' = V \cup \{\infty\}$ for some $U, V \in \mathcal{T}$. Then $U' \cap V' = (U \cup \{\infty\}) \cap (V \cup \{\infty\}) = (U \cap V) \cup \{\infty\} \in \mathcal{T}'$, and $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$. Since both $X \setminus U$ and $X \setminus V$ are compact and closed in X, the same is true for their union, hence again $U' \cap V' \in \mathcal{T}'$.

(T3) Suppose for each i in some indexing set I, that $U_i' \in \mathcal{T}'$. Then either $U_i' \in \mathcal{T}$ for all $i \in I$, in which case $\bigcup_{i \in I} U_i' \in \mathcal{T} \subseteq \mathcal{T}'$, or $U_j' = U_j \cup \{\infty\}$ for j in some non-empty subset $J \subseteq I$, and $X \setminus U_j$ is compact and closed in X. If $U_i' \in \mathcal{T}$ then $X \setminus U_i'$ is closed in X. Hence in this case $X' \setminus \bigcup_{i \in I} U_i'$ is a closed subset of X, and also a closed subset of a compact set $X \setminus U_j$, (for any $j \in J$) hence is compact. Hence again $\bigcup_{i \in I} U_i' \in \mathcal{T}'$.

Next we show that $(X, \mathcal{T})$ is a subspace of $(X', \mathcal{T}')$. First suppose that $U \subseteq X$ is in the topology induced on X by $\mathcal{T}'$. Then $U = U' \cap X$ for some $U' \in \mathcal{T}'$. Now either $U' \in \mathcal{T}$, in which case $U = U' \in \mathcal{T}$, or $U' = V \cup \{\infty\}$ where necessarily $V = U$ and $V \in \mathcal{T}$ says that $U \in \mathcal{T}$.

Conversely suppose that $U \in \mathcal{T}$ then $U = U \cap X$ is in the topology on X induced by $\mathcal{T}'$.

Finally we show that $(X', \mathcal{T}')$ is compact. Suppose that $\mathcal{U}$ is any open cover of X'. Then $\infty \in U_0$ for some $U_0 \in \mathcal{U}$. By construction, $X' \setminus U_0 = X \setminus U_0$ is compact, so there is some finite subcover $U_1, U_2, \ldots, U_n$ of $\mathcal{U}$ for $X \setminus U_0$. Then clearly $U_0, U_1, U_2, \ldots, U_n$ is a finite subcover of $\mathcal{U}$ for X'. So $(X', \mathcal{T}')$ is compact.

Additional exercise(s)

1. Suppose first that f is continuous. Then the graph of f is the inverse image of the diagonal $\Delta_Y \subset Y \times Y$ under the mapping $f \times \mathrm{id}_Y : X \times Y \to Y \times Y$; since Y is Hausdorff the diagonal is closed, and therefore the graph, which is the inverse image of the diagonal, must be closed. For this half of the proof, it is only necessary to assume that Y is Hausdorff.

Conversely, suppose that X and Y are compact Hausdorff and the graph $\Gamma_f \subset X \times Y$ is closed. Since X and Y are compact Hausdorff, a closed subset of their product is also compact Hausdorff. If $\varphi : \Gamma_f \to X$ is the restriction of the coordinate projection π_X to Γ_f , then φ is continuous, 1–1 and onto, and therefore by Proposition 13.26 in Sutherland the map φ is a homeomorphism onto its image. The map $\theta = \pi_Y | \Gamma_f$ is also continuous, and since $f = \theta \circ \varphi^{-1}$ as a map of sets, it follows that f is a composite of continuous maps and hence is continuous.■

2. Since p is a limit point of X, for each $M > 0$ there is some point $y \neq p$ such that $d(y, p) < 1/M$, and this implies that $f(y) > M$. Therefore f is not bounded on $X - \{p\}$.■

3. (i) In previous exercises we saw that $f(x) = d(x, B)$ is a continuous function of X. Furthermore, if B is a closed subset then $f(x) > 0$ on $X - B$. Since $A \subset X - B$ and A is compact, this means that $f|A$ must take a minimum value and this value must be strictly positive.■

(ii) Let B be the x-axis, and let A be the points on the hyperbola $xy = 1$ in the first quadrant. Then $d(x, B)$ is strictly positive on A since A and B are disjoint sets. However, we claim that the greatest lower bound for the set of values $\{d(x, B) \mid x \in A\}$ is equal to zero. This follows because the distance from $p_x = (x, 1/x)$ to $q_x = (x, 0)$ is $1/x$, which means that $d(p_x, B) < 1/x$. If we take x to be the positive integer n, then it follows that $d(p_n, B) < \frac{1}{n}$ for all n, which means that the greatest lower bound for the set of values is less than or equal to zero. Since 0 is a lower bound for the set of values, it follows that the greatest lower bound is zero, as claimed. However, as noted above the function $f(x)$ is always positive for $x \in A$, so $f(x)$ does not take a minimum value on A.■

4. By Exercise 11.2 in Sutherland, every one point set is closed in a Hausdorff space, and this is enough to prove that for each $A \subset X$ the set $L(A)$ of limit points is closed in X. (*Proof:* It suffices to show that $L(L(A)) \subset L(A)$. Given $p \in L(L(A))$ and an open neighborhood U of p we know that $(U - \{p\}) \cap L(A) \neq \emptyset$. Let y be a point in this intersection. Then $y \neq p$ and $V = U - \{y\}$ is also an open neighborhood of p, so we also have $(U - \{p, y\}) \cap A \neq \emptyset$. Since the left hand side is contained in $(U - \{p\}) \cap A$ it follows that p must be a limit point of A.)

Suppose now that X is Hausdorff and $\overline{A}$ is compact. Since $L(A)$ is contained in the closure and $L(A)$ is closed in X, it follows that $L(A)$ is a closed, hence compact, subset of $\overline{A}$.■

14. Sequential compactness

Exercises from Sutherland

`See the next four pages.`

Solutions to Chapter 14 exercises

14.1 Consider the sequence $(1/n)$ in $(0,\, 1)$. This has no subsequence converging to a point of $(0,\, 1)$ since the sequence $(1/n)$, and hence every subsequence, converges in $\mathbb{R}$ to 0.

14.2 Suppose for a contradiction that the sequentially compact metric space $(X,\, d)$ is not bounded. Choose any point $x_0 \in X$. Then for any $n \in \mathbb{N}$ there exists a point in X, call it x_n, with $d(x_n,\, x_0) \geqslant n$. The sequnce (x_n) has no convergent subsequence, since any subsequence (x_{n_r}) is unbounded ($d(x_{n_r},\, x_0) \geqslant n_r$). Hence X must be bounded.

14.3 Let A be a closed subset of a sequentially compact metric space X. Let (x_n) be any sequence in A. Then (x_n) is also a sequence in X, which is sequentially compact, so there is a convergent subsequence (x_{n_r}). The point this converges to must lie in A since A is closed in X (see Corollary 6.30). Hence A is also sequentially compact.

14.4 Let A be a sequentially compact subspace of a metric space X, and let $x \in \overline{A}$. Then (see Exercise 6.26) there is a sequence (a_n) of points in A converging to x. Since A is sequentially compact, there is some subsequence (a_{n_r}) of (a_n) converging to a point in A. But every subsequence of (a_n) converges to x, so $x \in A$. This tells us that A is closed in X (see Proposition 6.11 (c)).

14.5 Let (y_n) be a sequence in $f(X)$. For each $n \in \mathbb{N}$ there exists a point $x_n \in X$ such that $y_n = f(x_n)$. Since X is sequentially compact, there is some subsequence (x_{n_r}) of (x_n) which converges to a point $x \in X$. Then by continuity of f the subsequence $(y_{n_r}) = (f(x_{n_r})$ converges in Y to $f(x)$ (see Exercise 6.25). Hence $f(X)$ is sequentially compact.

14.6 This follows from Exercise 14.5. For if $f : X_1 \to X_2$ is a homeomorphism and X_1 is sequentially compact then so is X_2 by Exercise 14.5, since f is continuous and onto. Since the inverse of f is also continuous and onto, it follows likewise that if X_2 is sequentially compact then so is X_1.

14.7 This follows from Exercises 14.5 and 14.2. For if $f : X \to Y$ is a continuous map of metric spaces and X is sequentially compact, then by Exercise 14.5 so is $f(X)$, and hence, by Exercise 14.2, $f(X)$ is bounded.

14.8 By Exercise 14.7 the function f is bounded, so its bounds do exist. Now $f(X)$ is a sequentially compact subspace of $\mathbb{R}$ by Exercise 14.5. Hence $f(X)$ is closed in $\mathbb{R}$ by Exercise 14.4. But the bounds of a non-empty closed subset of $\mathbb{R}$ are in the set by Exercise 6.9. This says that the bounds of $f(X)$ are in $f(X)$, which means that they are attained.

14.9 Suppose that (X, d_X), (Y, d_Y) are sequentially compact metric spaces. In $X \times Y$ we shall use the product metric d_1: recall that $d_1((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$. Let $((x_n, y_n))$ be any sequence in $X \times Y$. First, since X is sequentially compact there is a subsequence (x_{n_r}) of (x_n) converging to a point $x \in X$. Now consider the sequence (y_{n_r}) in Y. Since Y is sequentially compact, there exists a subsequence $(y_{n_{r_s}})$ of (y_{n_r}) converging to a point $y \in Y$. Then $(x_{n_{r_s}})$ is a subsequence of (x_{n_r}) hence also converges to x. Consider the subsequence $((x_{n_{r_s}}, y_{n_{r_s}}))$ of $((x_n, y_n))$. This converges to (x, y): for let $\varepsilon > 0$. Since (x_{nr_s}) converges to x, there exists $S_1 \in \mathbb{N}$ such that $d_X(x_{nr_s}, x) < \varepsilon/2$ whenever $s \geqslant S_1$. Similarly there exists $S_2 \in \mathbb{N}$ such that $d_Y(y_{nr_s}, y) < \varepsilon/2$ whenever $s \geqslant S_2$. Put $S = \max\{S_1, S_2\}$. If $s \geqslant S$ then

$$d_1((x_{nr_s}, y_{n_{r_s}}), (x, y)) = d_X(x_{nr_s}, x) + d_Y(y_{nr_s}, y) < \varepsilon.$$

So $((x_n, y_n))$ has a subsequence converging to a point in $X \times Y$. This shows that $X \times Y$ is sequentially compact. (As we have seen, any 'product metric' will give the same answer.)

14.10 Suppose that the result is true for some $n \geqslant 1$, and let X be a bounded closed subset of $\mathbb{R}^{n+1}$. Then $X \subseteq [a, b]^{n+1}$ for some $a, b \in \mathbb{R}$, (by Exercise 5.7), and it is sufficient to prove that $[a, b]^{n+1}$ is sequentially compact, since X is closed in this space hence then also sequentially compact by Exercise 14.3. Now $[a, b]^n$ and $[a, b]$ are sequentially compact by inductive assumption and the allowed case $n = 1$ respectively, so $[a, b]^{n+1} = [a, b]^n \times [a, b]$ is sequentially compact by Exercise 14.9.

14.11 Let $x_n \in V_n$ for each $n \in \mathbb{N}$. Since X is sequentially compact, there is a subsequence (x_{n_r}) of (x_n) converging to some point $x \in X$. Since the V_n are nested, $x_{n_r} \in V_m$ for all r such that $n_r \geqslant m$. But V_m is closed in X, so $x \in V_m$ (by Corollary 6.30). This is true for all $m \in \mathbb{N}$, so $x \in \bigcap_{n=1}^{\infty} V_n$ and this intersection is non-empty.

14.12 Suppose that C is relatively compact in a metric space (X, d), and recall that for present purposes this means that $\overline{C}$ is sequentially compact. Now any sequence in C is also a sequence in $\overline{C}$, so it has a convergent subsequence. (In fact this subsequence converges to some point in $\overline{C}$).

Conversely suppose that every sequence in C has a convergent subsequence. We wish to show that $\overline{C}$ is sequentially compact. Let (x_n) be any sequence in $\overline{C}$. For each $n \in \mathbb{N}$, since $x_n \in \overline{C}$ there exists $y_n \in C$ such that $d(y_n, x_n) < 1/n$. Now consider the sequence (y_n) in C. By hypothesis this has a convergent subsequence (y_{n_r}), say converging to y. By Proposition 6.29, $y \in \overline{C}$. Now given any $\varepsilon > 0$ there exists $R_1 \in \mathbb{N}$ such that $d(y_{n_r}, y) < \varepsilon/2$ whenever $r \geqslant R_1$ and there exists $R_2 \in \mathbb{N}$ such that $1/n_r < \varepsilon/2$ whenever $r \geqslant R_2$. Put $R = \max\{R_1, R_2\}$. If $r \geqslant R$ then

$$d(x_{n_r}, y) \leqslant d(x_{n_r}, y_{n_r}) + d(y_{n_r}, y) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus any sequence in $\overline{C}$ has a subsequence converging to a point in $\overline{C}$ - in other words $\overline{C}$ is sequentially compact, so C is relatively compact.

14.13 The exercise does most of this! Following as suggested, we shall prove inductively that $[a, a_i] \subseteq A$ for $i = 1, 2, \ldots, a_n = b$. This is true for $i = 1$ since $a_0 = a \in A$, and since $a_1 - a_0 < \varepsilon$ where ε is a Lebesgue number for the cover $\{A, B\}$, we know that $[a_0, a_1]$ is contained in a single set of the cover, and this must be A since $A \cap B = \emptyset$. Suppose inductively that $[a, a_i] \subseteq A$ for some $i \in \{1, 2, \ldots, n-1\}$. Then we can repeat the above argument with a replaced by a_{n-1} and deduce that also $[a_{n-1}, a_n] \subseteq A$. Hence $[a, b] \subseteq A$, so $\{A, B\}$ is not a partition of $[a, b]$ after all. So $[a, b]$ is connected.

14.14 If $U_i = X$ for some $i \in \{1, 2, \ldots, n\}$ then any $\varepsilon > 0$ is a Lebesgue number for $\mathcal{U}$, since for any $\varepsilon > 0$, any set of diameter at most ε is contained in X and hence in U_i.

(i) Suppose now that $C_i \neq \emptyset$ for every $i \in \{1, 2, \ldots, n\}$. Then continuity of the function $f_i : X \to \mathbb{R}$ defined by $f_i(x) = d(x, C_i)$ follows from Exercise 6.16 (c). Also, from the definition it follows that all the values of $f_i(x)$ are non-negative.

(ii) Continuity of f follows from continuity of each f_i and Proposition 5.17. Let $x \in X$. Since $\mathcal{U}$ is a cover for X, $x \in U_i$ for at least one $i \in \{1, 2, \ldots, n\}$ so x is not in $C_i = X \setminus U_i$. Now C_i is closed in X, so $f_i(x) = d(x, C_i) > 0$ (by Exercise 6.16 (a)). But also $f_j(x) \geqslant 0$ for all $j \in \{1, 2, \ldots, n\}$ so $f(x) > 0$ as required.

(iii) By sequential compactness of X and Exercise 14.8, there exists $\varepsilon > 0$ such that $f(x) \geqslant \varepsilon$ for all $x \in X$.

(iv) Since there are just n values $d(x, C_i)$ it is clear that

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, C_i) \leqslant \max\{d(x, C_i) : i \in \{1, 2, \ldots, n\}\}.$$

(v) For a given $x \in X$ let $\max\{d(x, C_i) : i \in \{1, 2, \ldots, n\}\} = d(x, C_{k(x)})$. We prove that $B_\varepsilon(x) \subseteq U_{k(x)}$ where ε is as in (iii) above. For suppose $d(y, x) < \varepsilon$. Then $\varepsilon \leqslant f(x) \leqslant d(x, C_{k(x)})$ so $d(y, x) < d(x, C_{k(x)})$. This says $d(y, x)$ is less than the distance from x to $C_{k(x)} = X \setminus U_{k(x)}$, so $y \in U_{k(x)}$. Hence $B_\varepsilon(x) \subseteq U_{k(x)}$ as required. It follows that for any $x \in X$ there is a set $U \in \mathcal{U}$ such that $B_\varepsilon(x) \subseteq U$, so ε is a Lebesgue number for the cover $\mathcal{U}$.

14.15 If say V_{n_0} is empty, then $\bigcap_{n=1}^{\infty} V_n = \emptyset$, whose diameter is 0 by definition. Likewise in this case diam $V_{n_0} = 0$ so $\inf\{\text{diam } V_n : n \in \mathbb{N}\} = 0$ also.

Suppose now that all the V_n are non-empty. (We already know from Exercise 14.11 that their intersection is non-empty.) Now $\bigcap_{n=1}^{\infty} V_n \subseteq V_m$ for any $m \in \mathbb{N}$, so diam $\bigcap_{n=1}^{\infty} V_n \leqslant$ diam V_m. Hence diam $\left(\bigcap_{n=1}^{\infty} V_n\right) \leqslant \inf\{\text{diam } V_m : m \in \mathbb{N}\} = m_0$ say.

Conversely, m_0 is a lower bound for the diameters of the V_n, so for any $\varepsilon > 0$ and any $n \in \mathbb{N}$ we know that diam $V_n > m_0 - \varepsilon$. Hence there exist points $x_n, y_n \in V_n$ such that $d(x_m, x_n) > m_0 - \varepsilon$. Since X is sequentially compact, (x_n) has a subsequence (x_{n_r}) converging to a point $x \in X$, and then (y_{n_r}) has a subsequence $(y_{n_{r_s}})$ converging to a point $y \in X$. Since $(x_{n_{r_s}})$ is a subsequence of (x_{n_r}) it too converges to x. Also, by continuity of the metric, $d(x_{n_{r_s}}, y_{n_{r_s}}) \to d(x, y)$ as $s \to \infty$. Hence $d(x, y) \geqslant m_0 - \varepsilon$. Also, $x, y \in V_n$ for each $n \in \mathbb{N}$ since V_n is closed in X. Since this is true for all $n \in \mathbb{N}$, we have $x, y \in \bigcap_{n=1}^{\infty} V_n$. Hence diam $\bigcap_{n=1}^{\infty} V_n \geqslant m_0 - \varepsilon$. But this is true for any $\varepsilon > 0$, so diam $\bigcap_{n=1}^{\infty} V_n \geqslant m_0$.

The above taken together prove the result.

14.16(a) Any element of $\bigcap_{n=1}^{\infty} V_n$ must be in V_1, so it is the function f_m for some $m \in \mathbb{N}$. But $f_m \notin V_n$ for $n > m$. So (a) holds.

(b) For any two distinct elements f_l, f_m of V_n we know that $d_\infty(f_l, f_m) = 1$. This shows that diam $V_n = 1$.

(c) In this case, diam $\bigcap_{n=1}^{\infty} V_n = 0$, but $\inf\{\text{diam} V_n : n \in \mathbb{N}\} = 1$. So the conclusion of Exercise 14.15 fails. (We note that the space $\{f_n : n \in \mathbb{N}\}$ with the sup metric is not compact - see Example 14.23.)

14.17 (a) Let $x \in X$. We want to show that $x \in f(X)$. Consider the sequence (x_n) in X defined by:

$$x_1 = x, \;\; x_{n+1} = f(x_n) \;\text{ for all integers } n \geqslant 1.$$

Since X is sequentially compact, there is a convergent subsequence, say (x_{n_r}). Any convergent sequence is Cauchy, so given $\varepsilon > 0$ there exists $R \in \mathbb{N}$ such that $|x_{n_r} - x_{n_s}| < \varepsilon$ whenever $s > r \geqslant R$, in particular $|x_{n_R} - x_{n_r}|$ whenever $r > R$. Now we use the isometry condition, iterated $n_R - 1$ times, to see that $|x_1 - x_{n_r - n_R + 1}| < \varepsilon$ whenever $r \geqslant R$. But $x_1 = x$ and $x_{n_r - n_R + 1} \in f(X)$ whenever $r > R$. Hence $x \in \overline{f(X)}$. But X is compact and f is continuous, so $f(X)$ is compact. Also, X is metric hence Hausdorff, so $f(X)$ is closed in X. Hence $\overline{f(X)} = f(X)$. So $x \in f(X)$ for any $x \in X$, which says that f is onto. Hence f is an isometry.

(b) We can apply (a) to the compositions $g \circ f : X \to X$ and $f \circ g : Y \to Y$ to see that these are both onto. Since $g \circ f$ is onto, g is onto. Similarly since $f \circ g$ is onto, f is onto. Hence both f and g are isometries.

(c) We just define $f : (0, \infty) \to (0, \infty)$ by $f(x) = x + 1$.

1. (i) Assume that X is limit point compact, and let $\{x_n\}$ be an infinite sequence in X. If the set $S = \{x_0, x_1, \ etc.\}$ only has finitely elements, then some y in this set must appear infinitely often, and if we choose $n(k)$ so that $x_{n(k)} = y$ for each k, then $\{x_{n(k)}\}$ is a constant sequence and hence its limit is y. Suppose now that there are infinitely many distinct points in S. In this case we have a limit point y. Define a subsequence recursively starting with $x_{n(0)} = x_0$ by the usual method: If we have $x_{n(k)} \neq y$ for $k < m$, then we take $x_{n(m)}$ to be a point $x_p \neq y$ such that $p > n(m-1) > \ \cdots \ > n(m) = 0$ and $d(y, x_{n(m)})$ is less than $\frac{1}{m}$ and the minimum of $d(y, x_{n(k)})$ for $k < m$. It follows immediately that

$$\lim_{k \to \infty} x_{n(k)} \quad = \quad y$$

and hence we have constructed a convergent subsequence.

Conversely, assume that X is sequentially compact, and let A be an infinite subset of X. Then we can find a sequence $\{a_n\}$ in A such that $i \neq j$ implies $a_i \neq a_j$, and by sequential compactness this sequence has a convergent subsequence $\{a_{n(k)}\}$. If b is the limit of this sequence, we claim that $b \in L(A)$. Let $\varepsilon > 0$; since $d(b, a_{n(k)}) < \varepsilon$ for sufficiently large values of k, it suffices to show that we can construct a convergent subsequence such that $b \neq a_{n(k)}$ for all k. However, if $b = a_{n(m)}$ for some m, then we can obtain a new subsequence $\{c_{n(k)}\}$ with the desired property and the same limit by setting $c_{n(k)} = a_{n(k+m+1)}\}$ because $a_{n(k)} \neq a_{n(m)} = b$ for $k > m$.■

(ii) Follow the hint, and assume we have a subset S with no limit points. Then $\overline{S} = S \cup L(S) = S$, so S is closed in X, and likewise for every subset $T \subset S$. As indicated in the hint, take an infinite sequence of distinct points $x_k \in S$, let $T = \{x_1, x_2, \ etc.\}$, and set T_n equal to $T = \{x_n, x_{n+1}, \ etc.\}$. Then each T_n is a nonempty closed subset and $T_n \supset T_{n+1}$ for all n, but $\cap_n T_n = \emptyset$. This contradicts the alternate characterization of compactness in Exercise 13.6 of Sutherland (which also appears in `math145Anotes13.pdf`). The source of the contradiction was the assumption that $L(S) = \emptyset$, so this must be false. Therefore every infinite subset of the compact space X must have at least one limit point.■

2. As indicated in the hint, without loss of generality we may assume that $s < t$. Given a function f in $\mathcal{D}(A, B)$, the Mean Value Theorem implies that

$$f(t) \ - \ f(s) \quad = \quad f'(\xi) \cdot (t - s)$$

for some $\xi \in (s, t)$. By hypothesis $|f'(x)| \leq B$ for all x, and therefore $|f(t) - f(s)| = |f'(\xi)||t - s| \leq B \cdot |t - s|$. Therefore, if $\varepsilon > 0$ and $\delta = \varepsilon / B$, then $|s - t| < \delta$ implies $|f(s) - f(t)| < \varepsilon$.■